# Geometry

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## 1 Introduction

Geometry literally means *earth measure*. When geometry was developed by the ancients of various cultures, it was probably for the purposes of surveying, i.e., measuring the earth. In mathematics, geometry usually refers to a more general study of curves and surfaces. Different branches of geometry— differential geometry, algebraic geometry, geometric analysis— are distinguished in part by the objects of interest and in part by the different sets of tools brought to bear. In differential geometry, for example, the objects are curves and surfaces in complex space and the tools arise largely from differential calculus. Algebraic geometry is the study of objects that can be described using rational functions. The tools are typically, but not always, of an algebraic nature. Some of them are quite elaborate and rarely encountered by undergraduates, and only encountered by graduate students who will specialize in algebraic geometry or in an area that relies on algebraic geometry, for example computer science.

Mathematicians who study geometry usually do not talk much about similar triangles or alternate interior angles or angles in a circle or most of the other things you probably think of when you think "geometry." A part of Euclidean geometry that does come up in many branches of geometry studied today is *projective geometry.* Although this is not a course in projective geometry, we will study certain projective planes here. In that sense, this course forms a bridge from high school geometry to more advanced studies.

The high school geometry course you probably had was primarily a course in *synthetic* Euclidean geometry. Synthetic geometry starts from first principles: definitions, and *axioms* or *postulates*, which are truths held to be self-evident. The heart of the subject is logical synthesis applied to the axioms in a rigorous way to uncover the facts (*theorems*) which relate the objects of the geometry. This remains an attractive subject for students of any age with any background because all assumptions are, ideally, clearly stated at the beginning of the program. The objects are familiar and the relations among them rich. No previous knowledge is necessary.

Planar Euclidean geometry is the geometry of straight edge and compass. The straight edge is unmarked, so cannot measure length, and the compass does not stay open once you lift it from the page. The objects and all relations among them must be produced using only a straight edge and compass applied to a flat surface. One approach to studying Euclidean geometry is constructive, and usually a portion of the high school course deals with actual constructions. You can think of the synthetic approach as supplying detailed instructions for allowed constructions.

The book that brought synthetic geometry to us in the West was *The Elements* of Euclid, a masterwork dating from around the year 300 B.C. Not only has *The Elements* been studied more or less continuously since it was written, it has remained the most important source for the subject since that time. High school students usually study textbooks based on *The Elements*, not *The Elements* itself. These textbooks are, more or less, the products of interpretations, corrections, and "improvements" to *The Elements*— many of questionable merit — that have been incorporated into the subject since the time of Euclid. One view of these sorts of textbooks is that they attempt to provide the elusive "royal road to geometry,"[1] fit for schoolchildren.[2], p. vi.

Our course will start with *The Elements* itself, in a translation that dates to the beginning of the last century. Using this as a departure, we explore

[1]There is a legend that Ptolemy, the king of Egypt, asked Euclid, his tutor, if there was not a shorter way to learn geometry. Must one go through *The Elements*? Euclid's response was that there was "no royal road to geometry." Similar stories are attributed to other mathematicians in response to complaints from other kings trying to learn mathematics. See [2], p. 1.

some developments in Euclidean geometry since the glory days of Greece. Since a great deal has happened over the last few centuries, what you see here represents a very small sampling of topics taken from what is available for study under the heading 'geometry.'

Well before the twentieth century, mathematicians recognized that *The Elements* had gaps and inaccuracies. There were many significant contributions to rectifying *The Elements* but David Hilbert's *Grundlagen der Geometrie* (*Foundations of Geometry*), dating from the early 1900s, became the definitive correction and essentially closed the matter once and for all. Hilbert was able to control some of the difficulties with *The Elements* that had been recognized but had remained unresolved since antiquity. One of the more significant contributions of Hilbert was his recognition of the necessity of leaving certain terms undefined in an axiomatic development. Aristotle, who predated Euclid, thought a great deal about defining basic geometric objects and noted that " '*the* definitions' of point, line, and surface ... all define the prior by means of the posterior" [2], p. 155. In other words, for predeccesors of Euclid through the dawn of the 20th century, it was common but uneasy practice to work with definitions where, for instance, you might use lines to define points and points to define lines. Moreover, it was well-known that this was a logical subversion. Hilbert recognized that the solution to the problem was to jettison some definitions altogether. One must start with a minimal set of basic terms that remain undefined: their properties are then detailed as postulates. For example, we do not define 'point', 'line', or 'plane' but we understand the nature of these objects through the postulates, for example, two distinct points determine a unique line, and a line together with a point not on that line determine a unique plane. All other definitions, for example 'angle', are then carefully crafted using this basic set of undefined terms.

Our foray into synthetic geometry includes contributions from the Renaissance that amounted to the initiation of the study of the projective plane through the addition of ideal points to the Euclidean plane. Our study of Euclid and Hilbert culminates with the nine point circle.

If Euclidean planar geometry derives from straight-edge and compass constructions, affine geometry derives from straight-edge constructions alone. This sounds poorer than Euclidean geometry but we come into a wealth of ideas when we use linear algebra to develop a model for affine geometry. This is an extension of the familiar ideas of coordinate geometry and leads naturally to the three dimensional vector space model for a projective plane, an important tool in use today.

Projective geometry dates to the 16th century, thus predates many of the fundamental ideas that we think of as typical of the modern era in mathematics, such as sets and functions. Nonetheless, projective geometry has a strong flavor of modernity as it provides a setting in which curves and surfaces have few or no exceptional configurations. This gives projective geometry tremendous power which we exhibit with a brief study of plane algebraic curves. Algebraic curves can be described parametrically with polynomials. Plane curves, that is, those that lie in $\mathbb{R}^2$ or $\mathbb{C}^2$, necessitate the introduction of calculations and tools that are manageable but rich enough to give the reader an appreciation of some of the ideas handled in modern algebraic geometry. Our goal in this final section of the course is Bezout's Theorem, which is a description of the intersection of two curves in a plane.

# 2 Synthetic Geometry

## 2.1 Background on Euclid and *The Elements*

There are no contemporary accounts of Euclid's life but there are some details about Euclid that scholars have been able to cobble together indirectly and about which there is little controversy.

It is generally held that Euclid founded a school of mathematics in Alexandria, Egypt.[2] In particular, Archimedes, 287-212 B.C., sometimes described as the greatest mathematician who ever lived, studied in Alexandria at Euclid's school and cited Euclid's work in his own writing. It is clear then that Euclid predated Archimedes. On the other hand, *The Elements* has detailed references to the works of Eudoxus and Theaetetus. To have learned their work, Euclid would have to have gone to Plato's Academy.[2], p. 2. The Academy was established outside Athens around 387 B.C.[3] There is general agreement that Euclid was too young to have studied with Plato at the Academy, and too old to have taught Archimedes in Alexandria. This helps narrow down the dates for Euclid which are currently accepted as about 325-265 B.C. [6]

As a student at the Academy, Euclid would have been the product of

[2] Alexandria is named for Alexander the Great, who, as a child, studied with Aristotle.

[3] This is the origin of the word *academic*. *Academy* is actually the name of the place where Plato set up his institution. The Academy remained in use until 526, another astonishingly long-lived force in the intellectual world.

a rich tradition of careful thought, schooled in the writings and teachings of Plato, Aristotle, and the other Greek geometers. Aristotle, a student of Plato himself, suggested in his own writings that his students had sources which codified the principles of mathematics and science, including geometry, that were accepted at that time. These sources presumably would have been available to Euclid as well. In other words, Euclid's was not the first geometry text, even if we restrict attention to the West. This does not diminish its greatness but it is important to maintain perspective. One commonly accepted view of *The Elements* is that it pulled together what was known in geometry at the time, with the goal of proving that the five platonic solids are the only solutions to the problem of constructing a regular solid. There is room for doubt there, though, as several books of *The Elements* have nothing whatever to do with the construction of the platonic solids.[2], p. 2.

*The Elements*, which is in thirteen books, starts with a set of 23 definitions. We quote from [2], p. 153-154.

1. A **point** is that which has no part.
2. A **line** is a breadthless length.
3. The extremities of a line are points.
4. A **straight line** is a line which lies evenly with the points on itself.
5. A **surface** is that which has length and breadth only.
6. The extremities of a surface are lines.
7. A **plane** surface is a surface which lies evenly with the straight lines on itself.
8. A plane **angle** is the inclination to one another of two lines in a plane which meet one another and do not lie in a straight line.
9. When the lines containing the angle are straight, the angle is called **rectilineal.**
10. When a straight line set up on a straight line makes the adjacent angles equal to one another, each of the equal angles is **right**, and the straight line standing on the other is called a **perpendicular** to that on which is stands.

11. An **obtuse angle** is an angle greater than a right angle.

12. An **acute angle** is an angle less than a right angle.

13. A **boundary** is that which is an extremity of anything.

14. A **figure** is that which is contained by any boundary or boundaries.

15. A **circle** is a plane figure contained by one line such that all the straight lines falling upon it from one point among those lying within the figure are equal to one another.

16. And the point is called the **center** of the circle.

17. A **diameter** of the circle is any straight line drawn through the center and terminated in both directions by the circumference of the circle, and such a straight line also bisects the circle.

18. A **semi-circle** is the figure contained by the diameter and the circumference cut off by it. And the center of the semi-circle is that of the circle.

19. **Rectilineal figures** are those which are contained by straight lines, **trilateral figures** being those contained by three, **quadrilateral** those contained by four, and **multi-lateral** those contained by more than four straight lines.

20. Of trilateral figures, an **equilateral triangle** is that which has its three sides equal, an **isosceles triangle**, that which has two sides equal, and a **scalene triangle** that which has its three sides unequal.

21. Further, of trilateral figures, a **right-angled triangle** is that which has a right angle, and **obtuse-angled triangle** that which has an obtuse angle, and an **acute-angled triangle**, that which has three angles acute.

22. Of quadrilateral figures, a **square** is that which is both equilateral and right-angled; an **oblong** that which is not equilateral but right-angled; a **rhombus** that which is equilateral but not right-angled; and a **rhomboid** that which has its opposite sides and angles equal to one another but is neither equilateral nor right-angled. All other quadrilaterals are called **trapezia**.

23. **Parallel straight lines** are straight lines which, being in the same plane and being produced indefinitely in both directions, do not meet one another in either direction.

Mathematicians and philosophers succeeding Euclid left a wealth of commentary on the definitions, especially those of point, line, and plane. The definition of a line as the shortest distance between two points, for instance, is due to Legendre,[4] and dates to 1794.

The second and fourth definitions indicate a distinction between the notions of line and straight line. What Euclid calls a line, we usually call a curve. Euclid's definitions thus contain a classification of curves into three types: lines, circles, and everything else. That may not be so strange but it is worthwhile to note that more detailed classifications of curves were of great interest to geometers through the centuries, Euclid's predecessors and contemporaries included.

The definition of angle refers to lines, not necessarily straight lines. We are used to thinking of an angle as a "rectilineal angle," in Euclid's terminology. When we talk about the angle at which curves intersect, we may get a little nervous, and think something like, "The angle of intersection of curves at a point would be defined as the angle of intersection of their tangents at that point." Since we are raised from an early age on calculus, we may feel awkward because we know that curves may or may not have defined tangents at a particular point. Euclid is not concerned with this problem as, once again, he directs his energies towards lines and circles, which present no such ambiguities.

Another note is that Euclid's definition of a "rectilineal figure" at first seems to refer to what we would call a "polygon" but closer examination reveals a distinction. In standard usage today, a triangle is three noncollinear points and the line segments they determine. Euclid's definition of triangle refers to the space enclosed by what we call a triangle. The object Euclid called a triangle is today called a 2-simplex. What Euclid calls a circle, we would call a disk. In spite of this nicely detailed dictionary, make sure that you know what someone means when he says "circle" or "triangle." Context

[4]According to The MacTutor History of Mathematics Archive, http://www-history.mcs.st-andrews.ac.uk/history/index.html, Legendre's reworking of *The Elements*, "*Eléments de géométrie ...* was the leading elementary text on the topic for around 100 years" and actually formed the basis for most subsequent texts on Euclidean geometry in Europe and the United States.

often makes it clear but in any case, it is not a dumb question.

The definition of a circle is not what we are used to.

**Exercise**
What is the usual definition of a circle? Show that Euclid's definition is equivalent to the more familiar definition.

Euclid's five postulates follow the definitions. [2], p. 154-155

1. A straight line can be drawn from any point to any point.
2. A finite straight line can be produced continuously in a straight line.
3. One may describe a circle with any center and any radius.
4. All right angles are equal to one another.
5. (The Parallel Postulate) Consider a straight line falling on two straight lines. If the interior angles on the same side are less than two right angles, then the two straight lines, if produced indefinitely, meet on that side on which the angles are less than the two right angles.

Much later in *The Elements*, Euclid reveals that he actually assumes that there is a *unique* line determined by each pair of points. [2] p. 195. The second postulate says that a line segment (*finite straight line*) can be extended into a line. There is another hidden assumption there, as well, viz., that the line determined by a segment is unique.[2], p. 196.

The Parallel Postulate is not what we are used to seeing.

**Exercise**
What is the usual formulation for the existence of parallel lines in a plane? Show that this follows from Euclid's fourth and fifth postulates. (See [2], p. 220 for other formulations of the parallel postulate and the associated history.)

The fifth postulate was controversial from Euclid's time through succeeding centuries. The concern was whether it should really be a theorem, that is, whether the fifth postulate actually followed from the other four. A brief history of attempts to prove it is given in [2], p. 202-219. This is an important part of the history of mathematics partly because all the fussing over the

Parallel Postulate led to the development of noneuclidean geometries during the 19th century. Noneuclidean geometry may seem like an interesting idea that is of purely academic interest but nothing could be farther from the truth. Consider, for example, the geometry of the surface of a sphere. You realize how important this might be to humans when you think about traversing large distances across the surface of the earth. Other noneuclidean geometries are routinely used in the study of space— as in the cosmos. (See http://archive.ncsa.uiuc.edu/Cyberia/Cosmos/CosmosShape.html for example.)

The development of noneuclidean geometries by Lobachevsky in Russia and Bolyai in Hungary proved that the parallel postulate is independent of the other four. In other words, if you keep the other four postulates but excise or change the parallel postulate, you still get a viable axiomatic system. That was an enormous breakthrough, settling a controversy that persisted from Euclid's time, although it was many years before mathematicians accepted and understood the significance of noneuclidean geometries.

Euclid states five "common notions" in addition to the definitions and postulates. These are often referred to as "axioms" in the literature. (See p. 221 in [2].) We quote from [2], p. 155:

1. Things which are equal to the same thing are also equal to one another.
2. If equals be added to equals, the wholes are equal.
3. If equals be subtracted from equals, the remainders are equal.
4. Things which coincide with one another are equal to one another.
5. The whole is greater than the part.

The common notions look a bit strange because of where we stand in the history of the subject. To discuss them further, we should pause to discuss *equivalence relations*, an idea that is relatively modern.

Let $S$ be a set. A relation $\sim$ on $S$ is called an *equivalence relation* provided

1. $\sim$ is *reflexive*, i.e., $a \sim a$ for all $a$ in $S$;
2. $\sim$ is *symmetric*, i.e., $a \sim b$ implies $b \sim a$;
3. $\sim$ is *transitive*, i.e., if $a \sim b$ and $b \sim c$, then $a \sim c$.

Equality is the archetype for equivalence relations. Isomorphism is an algebraic equivalence relation. Homeomorphism is a topological equivalence relation. Congruence is a geometric equivalence relation. We are so used to thinking about congruence that it seems strange to have to say much about it but remember that equivalence relations per se were not familiar to Euclid and remember that he was working the subject from the most basic principles.

The first common notion says that equality is transitive. **Beware that in the Heath edition of Euclid, the word *equal* is used to mean *congruent*.** In modern English usage, we say angles $\alpha$ and $\beta$ are equal only if $\alpha$ and $\beta$ are different names for the same angle. Different angles that have the same measure are not equal, but congruent. This may seem like splitting hairs but in some sense it underlies aspects of *The Elements* that bothered Euclid himself.

The evaluation of Euclid's postulate system was to whether the defined terms, postulates, and common notions form a set of *complete*, *consistent*, and *independent* basic notions. We have mentioned Legendre, who was the most important commentator of his time and possibly for the next hundred years. The late nineteenth and early twentieth centuries saw major developments in Euclidean geometry, most notably with the works of Pasch, in 1882; Veronese, in 1891; and David Hilbert, in 1903 [4]. By the time later editions of Hilbert's *Grundlagen der Geometrie* were published, the stage was set, if not for a genuine revolution, then for a jump in the evolution of mathematics. The importance of axiomatization and abstraction in mathematics was explored more deeply than ever before. Hilbert's work was certainly an important impetus in this direction. This movement was manifested most obviously in *The Principia Mathematica* of Russell and Whitehead, who sought to reduce all of mathematics to formal logic and set theory.[9] Bourbaki, the name for a group of mathematicians (with changing membership) based in Paris from the 1930s through the present, initiated an ambitious project of putting all of mathematics on a firm axiomatic foundation. Whether or not one prefers this approach to mathematics— some argue that examples rather than axiomatization should drive the subject and that a focus on abstraction leads one astray of applications, a level of 'purity' that is fundamentally misguided— it was and remains a profound influence. Regardless of how you respond to the role of abstraction in mathematics, it has been hugely effective both in dispatching certain problems that remained intractable through the ages (for example, Fermat's Last Theorem) and in bringing certain subjects—

notably, algebraic geometry— to a better place for further development. (See [6] for more information about Bourbaki and their project. Texts that resulted from their work are still very much in use.)

We consider some of the results (*propositions*) Euclid proves in *The Elements* and some possible problems with the proofs.

Proposition 1, its proof, and the accompanying figure, we quote from [2], p. 241.

**Proposition 1.** *On a given finite straight line to construct an equilateral triangle.*

Let $AB$ be the given finite straight line.
Thus it is required to construct an equilateral triangle on the straight line $AB$.
With center $A$ and distance $AB$ let the circle $BCD$ be described; again, with center $B$ and distance $BA$ let the circle $ACE$ be described; and from the point $C$, in which the circles cut one another, to the points $A$, $B$ let the straight lines $CA$, $CB$ be joined.
Now, since the point $A$ is the center of the circle $CDB$, $AC$ is equal to $AB$.
Again, since the point $B$ is the center of the circle $CAE$, $BC$ is equal to $BA$.
But $CA$ was also proved equal to $AB$; therefore, each of the straight lines $CA$, $CB$ is equal to $AB$.
And things which are equal to the same thing are also equal to one another; therefore, $CA$ is also equal to $CB$.
Therefore the three straight lines $CA$, $AB$, $BC$ are equal to one another.
Therefore the triangle $ABC$ is equilateral; and it has been constructed on the given finite straight line $AB$.

(Being) what it was required to do.

Early commentators noticed that there is an unarticulated assumption used in this proof, namely that the circles must intersect. Extensive comments elaborating on this point are in [2], p. 242-3. The correction involves an assumption about the continuity of a line or curve.

We continue on to Proposition 2, again quoting statement, proof, and figure from [2], p. 244.

**Proposition 2.** *To place at a given point (as an extremity) a straight line equal to a given straight line.*

Let $A$ be the given point, and $BC$ the given straight line.
Thus it is required to place at the point $A$ (as an extremity) a straight line equal to the given straight line $BC$.

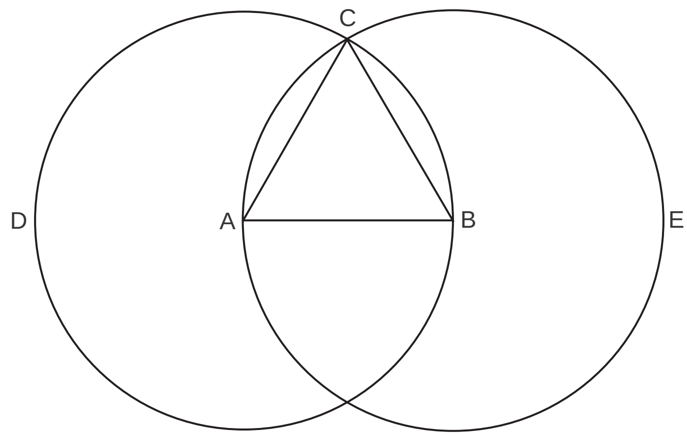


Figure 1: Proposition 1

From the point $A$ to the point $B$ let the straight line $AB$ be joined; and on it let the equilateral triangle $DAB$ be constructed.
Let the straight lines $AE$, $BF$ be produced in a straight line with $DA$, $DB$; with center $B$ and distance $BC$ let the circle $CGH$ be described; and again, with center $D$ and distance $DG$ let the circle $GKL$ be described.
Then, since the point $B$ is the center of the circle $CGH$, $BC$ is equal to $BG$.
Again, since the point $D$ is the center of the circle $GKL$, $DL$ is equal to $DG$.
And in these $DA$ is equal to $DB$; therefore the remainder $AL$ is equal to the remainder $BG$.
But $BC$ was also proved equal to $BG$; therefore each of the straight lines $AL$, $BC$ is equal to $BG$.
And things which are equal to the same thing are also equal to one another; therefore $AL$ is also equal to $BC$.
Therefore at the given point $A$ the straight line $AL$ is placed equal to the given straight line $BC$.

(Being) what it was required to do.

The statement of the proposition is that, given a point $A$ and a line segment $BC$, we can construct a line segment $AL$ which is congruent to $BC$.

Euclid's first step is to employ Proposition 1 to construct an equilateral triangle $DAB$. The points $E$ and $F$ are respectively points on the lines determined respectively by the segments $DA$ and $DB$. Euclid is invoking

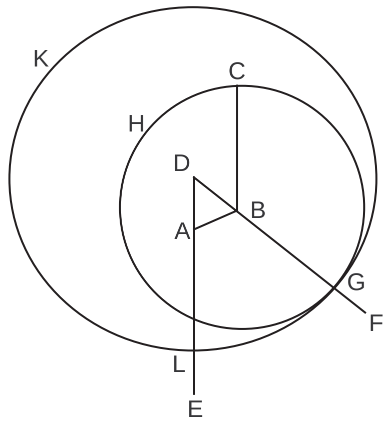


Figure 2: Proposition 2

the second postulate here, which says, in our language, that a line segment is part of a line.

Next, Euclid constructs two circles, the first with center $B$, radius $BC$. $G$ and $H$ are points on this circle, but note that while $H$ is arbitrary, $G$ must be the point where this circle intersects the line $BD$, as in the picture. The second circle has center $D$, radius $DG$, and contains points $K$ and $\ell$ where $\ell$ is the intersection point of the line $AD$ and the circle. The circle centered at $B$ we now call $CGH$; the circle centered at $D$ we now call $GKL$. As radii of the same circle, $BC \cong BG$. Similarly, $DL \cong DG$. By construction, $DA \cong DB$. The next step invokes Common Notion 3, which allows us to subtract line segments, essentially, to get $AL \cong BC$.

Try to answer the following questions. Does a correct proof of Proposition 2 depend on the relative positions of the point $A$ and the segment $BC$? For instance, if $A$ is on the line determined by $BC$, does the proof still apply? Does it change depending on whether $A$ is between $B$ and $C$? What if $A$ is an endpoint of the segment? In this case, we ought to be able to construct a segment extending off the given segment with the same length. Does Euclid's proof apply in that case? If the proof needs modification, do you think Euclid missed something or do you think he made a deliberate choice?

If you think about constructing a segment of a given length, using a straight edge and compass, you would probably mark the endpoints of the given segment with the compass, move the compass to a new location, mark

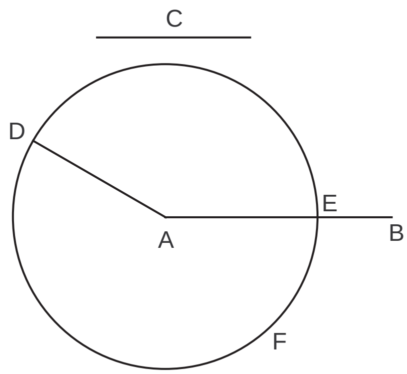


Figure 3: Proposition 3

the endpoints of the new segment, and use the straight edge to connect the dots. This violates the stricture that the compass used for Euclidean geometry does not stay open when we lift it off the page. Proposition 2 proves that even with a floppy compass, you can transfer information about length from one location to another. Proposition 3 is a corollary that says we can cut a given line segment to the match the length of a smaller segment.

We quote from [2], p. 246.

**Proposition 3.** *Given two unequal straight lines, to cut off from the greater a straight line equal to the smaller.*

Let $AB$, $C$ be the two given unequal straight lines and let $AB$ be the greater of them. Thus it is required to cut off from $AB$ the greater a straight line equal to $C$ the less. At the point $A$ let $AD$ be placed equal to the straight line $C$; and with center $A$ and distance $AD$ the circle $DEF$ be described. Now, since the point $A$ is the center of the circle $DEF$, $AE$ is equal to $AD$. But $C$ is also equal to $AD$. Therefore each of the straight lines $AE$, $C$ is equal to $AD$; so that $AE$ is also equal to $C$. Therefore, given the two striaght lines $AB$, $C$, from $AB$ the greater $AE$ has been cut off equal to $C$ the less. (Being) what it was required to do.

Again, look for implicit assumptions about the relative positions of objects and ask if they make a material difference to the proof.

The next two propositions are the first substantial theorems about triangles. Each has spawned much comment through the centuries. Proposition 4 is the familiar side-angle-side criterion for triangle congruence. We quote from [2], p. 247-8. This time, we omit Euclid's figure. You should supply one as an aid to follow the argument.

**Proposition 4.** *If two triangles have the two sides equal to two sides respectively, and have the angles contained by the equal straight lines equal, they will also have the base equal to the base, the triangle will be equal to the triangle, and the remaining angles will be equal to the remaining angles respectively, namely those which the equal sides subtend.*

Let $ABC$, $DEF$ be two triangles having the two sides $AB$, $AC$ equal to the two sides $DE$, $DF$ respectively, namely $AB$ to $DE$ and $AC$ to $DF$, and the angle $BAC$ equal to the angle $EDF$.
I say that the base $BC$ is also equal to the base $EF$, the triangle $ABC$ will be equal to the triangle $DEF$, and remaining angles will be equal to the remaining angles respectively, namely those which the equal sides subtend, that is, the angle $ABC$ to the angle $DEF$, and the angle $ACB$ to the angle $DFE$.
For, if the triangle $ABC$ be applied to the triangle $DEF$, and if the point $A$ be placed on the point $D$ and the straight line $AB$ on $DE$, then the point $B$ will also coincide with $E$, because $AB$ is equal to $DE$.
Again, $AB$ coinciding with $DE$, the straight line $AC$ will also coincied with $DF$, because the angle $BAC$ is equal to the angle $EDF$; hence the point $C$ will also coincide with the point $F$, because $AC$ is again equal to $DF$.
But $B$ also coincided with $E$; hence the base $BC$ will coincide with the base $EF$.
[For if, when $B$ coincides with $E$ and $C$ with $F$, the base $BC$ does not coincide with the base $EF$, two straight lines will enclose a space: which is impossible. Therefore the base $BC$ will coincide with $EF$] and will be equal to it.
And the remaining angles will also coincide with the remaining angles and will be equal to them, the angle $ABC$ to the angle $DEF$, and the angle $ACB$ to the angle $DFE$.
Therefore etc.

(Being) what it was required to prove.

The controversy about the proof is to do with the so-called "method of superposition," that is, the business of picking up one triangle and dropping it down onto another for comparison. Heath et. al. aver that Euclid himself was uncomfortable about employing this argument [2], p. 249. In fact, Heath argues that Common Notion 4 and the bracketed material in the proof were added by later commentators to justify this step in Euclid's proof. [2], p. 225.

Why is superposition problematic? The argument, which goes back to the 16th century, is that if superposition is indeed a legitimate method of proof, one could use it to prove many of the propositions in *The Elements* with little trouble. Consider the pains Euclid took to prove Proposition 2. If superposition were allowed, we could use it there and dispatch the proof in one line. Using it in this proof and not the others belies a conflict in the underlying ideas. Hilbert resolves the conflict with a dramatic flourish, which we take up in the next section. (See [2], pp. 249-250 for more details regarding this and other difficulties with the proof of Proposition 4.)

We will grant Proposition 4 and proceed to consider Proposition 5, known famously as *pons asinorum*, the asses' bridge.[1], p. 6. The most common stories behind the name are (1) that it refers to those who cannot follow the proof , or (2) that it refers to the picture Euclid supplied for his proof. The actual origin and meaning of the name is lost, but people still enjoy puzzling over it. [2], p. 415

We quote the statement, proof, and figure from [2], p. 251.

**Proposition 5.** *In isosceles triangles the angles at the base are equal to one another, and, if the equal straight lines be produced further, the angles under the base will be equal to one another.*

Let $ABC$ be an isosceles triangle having the side $AB$ equal to the side $AC$; and let the straight lines $BD$, $CE$ be produced further in a straight line with $AB$, $AC$.
I say that the angle $ABC$ is equal to the angle $ACB$, and the angle $CBD$ to the angle $BCE$.
Let a point $F$ be taken at random on $BD$; from $AE$ the greater let $AG$ be cut off equal to $AF$ the less; and let the straight lines $FC$, $GB$ be joined.
Then, since $AF$ is equal to $AG$ and $AB$ to $AC$, the two sides $FA$, $AC$ are equal to the two sides $GA$, $AB$ respectively; and they contain a common angle, the angle $FAG$.
Therefore the base $FC$ is equal to the base $GB$, and the triangle $AFC$ is equal to the triangle $AGB$, and the remaining angles will be equal to the remaining angles respectively, namely those which the equal sides subtend, that is, the angle $ACF$ to the angle $ABG$, and the angle $AFC$ to the angle $AGB$. And, since the whole $AF$ is equal to the whole $AG$, and in these $AB$ is equal to $AC$, the remainder $BF$ is equal to the remainder $CG$.
But $FC$was also proved equal to $GB$; therefore the two sides $BF$, $FC$ are equal to the two sides $CG$, $GB$ respectively; and the angle $BFC$ is equal to the angle $CGB$, while the base $BC$ is common to them; therefore the triangle $BFC$ is also

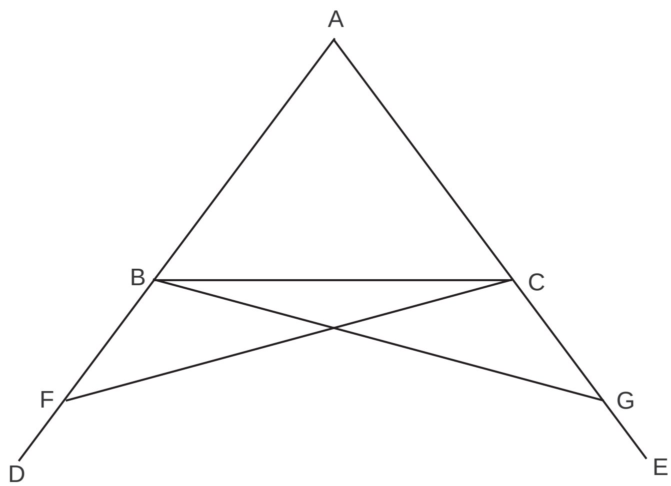


Figure 4: Pons Asinorum

equal to the triangle $CGB$, and the remaining angles will be equal to the remaining angles respectively, namely those which the equal sides subtend; therefore the angle $FBC$ is equal to the angle $GCB$, and the angle $BCF$ to the angle $CBG$.
Accordingly, since the whole angle $ABG$ was proved equal to the angle $ACF$, and in these the angle $CBG$ is equal to the angle $BCF$, the remaining angle $ABC$ is equal to the remaining angle $ACB$; and they are at the base of the triangle $ABC$. But the angle $FBC$ was also proved equal to the angle $GCB$; and they are under the base.
Therefore etc.

Q.E.D.

*Pons asinorum* is significant for several reasons. First, it can be argued that this is the board from which a great many of the fundamental theorems of Euclidean geometry spring. This is, in fact, Coxeter's approach to teaching geometry in [1] (cf. Section 1.3.) Second, references in Aristotle indicate that geometers who preceded Euclid knew *pons asinorum* but had a different proof, and indeed, quite a different approach to the subject altogether. This helps us understand the role of *The Elements* in history. Euclid was, to a large extent, pulling together known results but from early on in the program, he forges his own way through the thicket. See [2], pp. 252-254. Finally, Proclus, a prominent commentator who came around 700 years after Euclid, relates the proof that Pappus described, which is quite possibly what you saw in high school. (Compare also [1], Section 1.3.) Here is Pappus's proof, as quoted from [2], p. 254. Note that this does not cover the part of the theorem describing the angles under the base of the triangle.

Let $ABC$ be an isosceles triangle, and $AB$ equal to $AC$. Let us conceive this one triangle as two triangles, and let us argue in this way. Since $AB$ is equal to $AC$, and $AC$ to $AB$, the two sides $AB$, $AC$ are equal to the two side $AC$, $AB$. And the angle $BAC$ is equal to the angle $CAB$ for it is the same. Therefore all the corresponding parts (in the triangles) are equal, namely $BC$ to $BC$, the triangle $ABC$ to the triangle $ABC$ (i.e. $ACB$), the angle $ABC$ to the angle $ACB$, and the angle $ACB$ to the angle $ABC$, for these are the angles subtended by the equal sides $AB$, $AC$. Therefore in isosceles triangles the angles at the base are equal.

Pappus's proof is often presented without attribution and with modification, namely, that the triangle is picked up from the page and reflected across a median or angle or base bisector (none of which have been defined) and dropped down again upon itself...or upon a trace it has left on the page after it has been picked up.... In light of previous remarks, this approach is at best unsettling and Pappus's original approach, to "conceive this one triangle as two triangles," while faintly suspect, seems elegant in comparison. Before leaving this topic, we invoke a rebuke to commentators who proposed to improve Euclid's proof by means such as these. This is a quote of C. L. Dodgson (a.k.a. Lewis Carroll) from *Euclid and His Modern Rivals*, as cited in [1], p. 6: "MINOS: It is proposed to prove [pons asinorum] by taking up the isosceles Triangle, turning it over, and then laying it down again upon itself.

"EUCLID: Surely that has too much of the Irish Bull about it, and reminds one a little too vividly of the man who walked down his own throat, to deserve a place in a stricly philosophical treatise?"

Book I of Euclid's *Elements* ends with Proposition 48, the theorem that says that if the squares of two sides of a triangle add up to the square of the third side, then it is a right triangle. Proposition 47 is the Pythagorean Theorem. Note that a square here is a geometric figure. In these books of *The Elements*, Euclid does not work with numbers. Book II is concerned with parallelograms and Book III takes up circles. Our goal for this part of the course is the nine point circle but for now we turn away from Euclid to his most successful modern rival, Hilbert, so that we can proceed to our goal on a more secure foundation.

## 2.2 Hilbert's Axioms

Hilbert's goal in studying *The Elements* was to sort out the objects Euclid studied, the assumptions about them, and the relationships among them. His goal was a description of the foundation of Euclidean geometry that would yield all the familiar theorems while meeting modern standards of rigor. In his time, Euclid likely set the standard for rigor but standards change and Hilbert wanted to bring Euclid's work up to a modern level of acceptability.

Several of Euclid's definitions had to be jettisoned. By Hilbert's time, it had long been recognized that is was impossible to explain something like a straight line "by any regular definition which does not introduce words already containing in themselves, by implication, the notion to be defined." [2], p. 168 (cf. Aristotle's remark about defining the prior with the posterior.) Predecessors of Hilbert started with material approximations to terms like *point* and *line*. Hilbert was the one responsible for acknowledging the necessity of starting with terms that were and would remain, frankly, undefined. This may seem like a triviality but it was an important idea in the early development of logic and set theory and, in some sense, goes back to Kant, whom Hilbert quotes before the introduction of *Foundations of Geometry.* [4], p. 2.

All human knowledge thus begins with intuitions, proceeds thence to concepts and ends with ideas.

Kant, *Critique of Pure Reason,* "Elements of Transcendentalism," *Second Part*, II.

Hilbert singled out the following undefined terms referring to objects: *points*, denoted $A$, $B$, $C$, etc., *lines*, denoted $a, b, c$, etc., *planes*, denoted $\alpha, \beta, \gamma$, etc, and *space.* The points are called the *elements of line geometry.* Points and lines are the *elements of plane geometry.* Points, lines and planes are the *elements of space geometry.* The undefined terms referring to relations among points, lines, planes, and spaces, are *lies on*, *between*, and *congruent.*

Hilbert's convention, which we follow, is that when we refer to points and lines with language such as, "$A$ and $B$ are points" it is to be understood that $A$ and $B$ are distinct.

Hilbert's axioms are organized into five subsets: Axioms of Incidence, Axioms of Order, Axioms of Congruence, Axiom of Parallels, and Axioms of Continuity. These are not verbatim but are only slightly modified from the translation of *Grundlagen* [4] that we are using.

**I. Axioms of Incidence**

1. Two points $A$ and $B$ determine a line $a$ that contains them.

2. The line determined by two points is unique.

We denote by $\overline{AB}$ the unique line determined by points $A$ and $B$.

3. Every line contains at least two points. There exist three points that do not lie on a single line.

This axiom is a guarantee against triviality, in other words, that points, lines, and planes are all different sorts of objects. Points that lie on a single line are *collinear.* Points that do not lie on a single line are *noncollinear.* Hilbert did not use these words in his axioms but did later on in the text.

4. Every set of three noncollinear points is contained in a plane. Every plane contains at least one point.

5. The plane determined by the three noncollinear points is unique.

6. If two points of a line $a$ lie in a plane $\alpha$, then all of the points of $a$ lie in $\alpha$.

7. If two planes have a point in common, then they have at least one more point in common.

8. There exist at least four points which do not lie in a plane.

Points that lie in a single plane are *coplanar*. Points that are not in a single plane are *noncoplanar.*

Modern set theory dates from the late 19th century with the work of Georg Cantor. Hilbert knew set theory but did not use its language and conventions in the *Grundlagen.* Although we will not deal with it in a deep way, we are so used to its basic ideas and notation that it seems natural to use them here. The underlying idea is that lines and planes are sets. The elements of these sets are points. If $A$ is a point lying on the line $a$ or the plane $\alpha$, we write $A \in a$ or $A \in \alpha$. If the line $a$ lies in the plane $\alpha$, we think of $a$ as a subset of $\alpha$ and write $a \subset \alpha$. If the point $A$ belongs to the lines $a$ and $b$, the line $a$ and plane $\alpha$, or the planes $\alpha$ and $\beta$, write $A \in a \cap b$, $A \in a \cap \alpha$, or $A \in \alpha \cap \beta$.

**Exercises**
Using only Hilbert's axioms as stated above, prove the following theorems.

1. The intersection of two lines in a plane is a single point or empty.
2. The intersection of two planes is empty or exactly one line and no other points.
3. Let $\alpha$ be a plane and $a$ a line that does not lie in $\alpha$. The intersection of $a$ and $\alpha$ is empty or a single point.
4. Let $a$ be a line and $A$ a point not on $a$. There is a exactly one plane containing $a$ and $A$.
5. Let $a$ and $b$ be lines with a single point of intersection. There is exactly one plane containing both $a$ and $b$.

**II. Axioms of Order**

Line segments and angles are not undefined terms but the notion of *betweenness* is and it underlies a great many of the familiar ideas from Euclidean geometry, in particular, segments and angles. Hilbert attributes this set of axioms to Pasch, a German mathematician of the late 19th centery. [4]

When the point $B$ lies between points $A$ and $C$, we write $A * B * C$. Designate the line determined by points $A$ and $B$ by $\overline{AB}$.

1. If $A * B * C$ then $A$, $B$, $C$ are collinear and $C * B * A$.
2. If $A$ and $C$ are two points there exists a point $B \in \overline{AC}$ with $A * C * B$.

Hilbert defines the *line segment* $AB$ or $BA$ to be the point $A$ and $B$ together with the points between $A$ and $B$. When $A * C * B$, $C$ is *inside* $AB$ otherwise $C$ is *outside* $AB$. $A$ and $B$ are the *endpoints* of $AB$. Note that Hilbert does not use the notion of *sensed* or *directed* line segments.

3. Of any three points on a line, there is no more than one that lies between the other two.

4. Let $A$, $B$, $C$ be noncollinear points and let $a$ be a line in the plane $ABC$ that does not contain any of $A$, $B$, or $C$ but that passes through $AB$. Then $a$ must also pass through $AC$ or $BC$.

The last axiom says that in a plane, the sides of a triangle are like doors to its interior and if a line enters through one door, it must exit through another.

**Exercise**
Use Axioms of Incidence and Betweenness to prove that for any points $A$ and $B$ there is a point $C$ on $AB$ with $A * C * B$. The argument makes use of Axiom II.4 and is not as easy as one might expect.

The axioms to this point allow us to prove several things that seem intuitively obvious when we talk about *space*. For us, space typically means $\mathbb{R}^n$, but remember that neither Euclid nor Hilbert was discussing $\mathbb{R}^n$. Hilbert in fact remarked famously that his axiom system should be as applicable to chairs, tables, and bottles of beer, as to points, lines, and planes. [8]

Omitting proofs, we state some of the consequences of the Axioms of Order.

**Theorem 1.** *Given any finite number of points on a line, it is possible to order them* $A, B, C, \dots, K$ *so that* $A * B * C$, $B * C * D, \dots, I * J * K$.

**Theorem 2.** *Between any two points on a line there exist an infinite number of points.*

**Theorem 3.** *Let* $a$ *be a line and* $\alpha$ *a plane containing* $a$. *The points of* $\alpha$ *that do not belong to* $a$ *fall into two collections, designated as sides of the line. For points* $A$ *and* $B$ *on the same side of* $a$, *the segment* $AB$ *does not intersect* $a$. *For points* $A$ *and* $C$ *on opposite sides of the line,* $AC$ *intersects* $a$.

As a line divides a plane into two sets, a point divides a line into two sets, and a plane divides 3-space into two sets. The Axioms of Order are what allow us to view polygons as having insides and outsides, for example. They also give us the notion of a *ray*.

**Definition 1.** *Let $A$, $A'$, $O$, $B$ be four points of the line $a$ such that $O$ lies between $A$ and $B$ but not between $A$ and $A'$. The points $A$, $A'$ are then said to lie on the line $a$* on one and the same side of $O$ *and the points $A$, $B$ are said to lie on the line $a$ on different sides of the point $O$. The totality of points of $a$ that lie on one and same side of $O$ is called a* ray *emanating from $O$. Thus every point of a line partitions it into two rays.*

If we are to have triangles, we need angles and some axioms guiding us in how to think about the notion of congruence.

**III. Axioms of Congruence**

When discussing segments and angles we use the words congruent and equal interchangeably.

The first axiom of congruence allows the possibility of constructing congruent segments. Recall that this was the second proposition in Book I of *The Elements*.

1. Given a segment $AB$ on a line $a$ and a line $a'$ with a point $A' \in a'$, we can find a point $B'$ on either side of $A'$ so that $AB \cong A'B'$.

The second axiom is a form of the first of Euclid's Common Notions.

2. If $AB \cong CD$ and $EF \cong CD$ then $AB \cong EF$.

**Exercise**
Prove that every segment is congruent to itself, i.e., that segment congruence is reflexive. Prove that segment congruence is also symmetric and transitive.

3. On the line $a$ let $AB$ and $BC$ be segments which, except for $B$, have no point in common. On the same or another line $a'$ let $A'B'$ and $B'C'$ be two segments which except for $B'$ also have no point in common. In that case, if $AB \cong A'B'$ and $BC \cong B'C'$ then $AC \cong A'C'$.

This axiom allows the addition of segments, as did Euclid's Common Notion 3.

**Definition 2.** *Let $\alpha$ be a plane and $h$, $k$ any two distinct rays emanating from $O$ in $\alpha$ and lying on distinct lines. The pair of rays $h$, $k$ is called an* angle *and is denoted $\angle(h,k)$ or $\angle(k,h)$.*

Straight angles and angles that exceed straight angles are excluded by this definition.[5] When necessary, we use $\overline{a}$ to distinguish a line $\overline{a}$ from a ray $a$ contained in $\overline{a}$.

An angle partions the points of a plane into the interior of the angle and the exterior of the angle.

**Exercises**

1. Give a precise definition of the interior and the exterior of an angle. Show that neither set is empty.

2. If $R$ is any region of a plane with the property that whenever points $A$ and $B$ are in $R$, then the segment $AB$ is entirely contained in $R$, then $R$ is *convex*. Show that the interior of an angle is convex. Is the exterior convex as well?

3. Let $H$ be a point of ray $h$ and $K$ be a point of the ray $k$ where $h$ and $k$ form an angle $\angle(h,k)$. Show that $HK$ lies in the interior of $\angle(h,k)$.

4. A ray emanating from $O$ lies either entirely inside or entirely outside the angle. A ray that lies in the interior intersects $HK$.

We continue with axioms of congruence.

4. Every angle in a given plane can be constructed on a given side of a given ray in a uniquely determined way. Moreover, every angle is congruent to itself.

As with segments, we are not treating angles as having sense or direction. Thus $\angle(h,k)$ and $\angle(k,h)$ denote the same angle.

5. If $\Delta ABC$ and $\Delta A'B'C'$ are triangles with $AB \cong A'B'$, $AC \cong A'C'$, and$\angle BAC \cong \angle B'A'C'$, then the congruence $\angle ABC \cong \angle A'B'C'$ is also satisfied.

Notice that Axiom III.5 also gives us $\angle BCA \cong B'C'A'$.

[5] In [4], it says that "obtuse angles" are excluded by this definition. This may be an error in the translation.

**Exercise**
Prove the uniqueness of segment construction, using uniqueness of angle construction and Axiom III.5. Start with the assumption that a segment congruent to $AB$ is constructed two ways on a ray emanating from $A'$ to $B'$ and to $B''$.

**Definition 3.** *Two angles with a common vertex and a common side are* supplementary *provided their other sides form a line. A* right angle *is an angle congruent to one of its supplements.*

Recall that Euclid defined a right angle in terms of lines impinging on one another. A bit later in Book I, there is a proposition that says a right angle is congruent to its supplement. Once again, then, we see Hilbert taking one of Euclid's propositions as an axiom.

Axiom III.5 does not quite give us the side-angle-side (SAS) criterion for triangle congruence but it is very close. Before settling the congruence theorems for triangles, we revisit pons asinorum.

**Theorem 4.** *The base angles of an isosceles triangle are congruent.*

*Proof.* Given isosceles $\Delta ABC$ with $AB \cong BC$, we can write $AB \cong BC$, $\angle B \cong \angle B$ (III.4), $BC \cong AB$. By III.5, $\angle A \cong \angle C$. □

**Theorem 5 (SAS or first congruence theorem for triangles).** *If* $\Delta ABC$ *and* $\Delta A'B'C'$ *satisfy* $AB \cong A'B'$, $\angle ABC \cong \angle A'B'C'$, $BC \cong B'C'$, *then* $\Delta ABC \cong \Delta A'B'C'$.

*Proof.* Axiom III.5 implies $\angle BAC \cong B'A'C'$ and $\angle ACB \cong A'B'C'$. It remains to show $AC \cong A'C'$.

Choose $C''$ on $\overline{A'C'}$ so that either $A' {*} C' {*} C''$ or $A' {*} C'' {*} C'$ and $A'C'' \cong AC$. We have shown that $C''$ is uniquely determined.

Note then that we have SAS for triangles $\Delta ABC$ and $\Delta A'B'C''$ since $AC \cong A'C''$, $\angle A \cong \angle A'$, and $AB \cong A'B'$. Axiom III.5 implies that $\angle ABC \cong A'B'C''$. We also have $\angle ABC \cong \angle A'B'C$". This violates uniqueness of angle construction as given in Axiom III.4, unless $C' = C''$. □

The next theorem is commonly called angle-side-angle (ASA).

**Theorem 6 (ASA or second congruence theorem for triangles).** *If* $\angle A \cong \angle A'$, $AB \cong A'B'$ *and* $\angle B \cong \angle B'$, *then* $\Delta ABC \cong \Delta A'B'C'$.

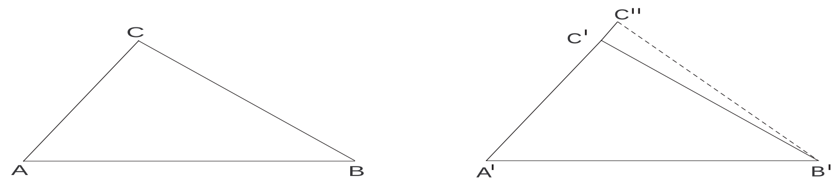


Figure 5: SAS for triangles

*Proof.* The picture above applies here as well. Choose $C''$ exactly as in the last proof and note that this gives us SAS for $\Delta ABC$ and $\Delta A'B'C''$ as $AC \cong A'C''$, $\angle A \cong \angle A'$ and $AB \cong A'B'$. The last theorem gives us $\Delta ABC \cong \Delta A'B'C''$ which in turn implies that $\angle ABC \cong \angle A'B'C''$. By assumption, $\angle ABC \cong \angle A'B'C'$ so that, again, unless $C' = C''$, uniqueness of angle construction is violated. We conclude that $AC \cong A'C'$, thus that $\Delta ABC \cong \Delta A'B'C'$ by SAS. $\square$

**Theorem 7.** *Supplements of congruent angles are congruent.*

*Proof.* Suppose $\angle ABC \cong \angle A'B'C'$. Without loss of generality, we can assume $AB \cong A'B'$ and $BC \cong B'C'$. By SAS, $\Delta ABC \cong A'B'C'$.

Now choose points $D$, $D'$ with $A * B * D$, $A' * B' * D'$, $BD \cong B'D'$. Again by SAS, $\Delta ADC \cong \Delta A'D'C'$. This gives us $\angle ADC \cong \angle A'D'C'$, in turn implying that $CD \cong C'D'$, thus SAS for $\Delta CBD$ and $C'B'D'$.

We conclude that $\angle DBC \cong \angle D'B'C'$, as desired. $\square$

**Corollary 1.** *Vertical angles are congruent.*

*Proof.* Consider the angles in Figure 7. Each angle marked with double arcs is supplementary to the one marked with a single arc. The latter is congruent to itself by Axiom III.4. By the previous theorem, the supplements are thus congruent. Now notice that we can treat any pair of vertical angles as supplements to a given angle which is congruent to itself. $\square$

The next lemma allows us to add and subtract congruent angles, thus to prove the side-side-side (SSS) criterion for triangle congruence. The reader

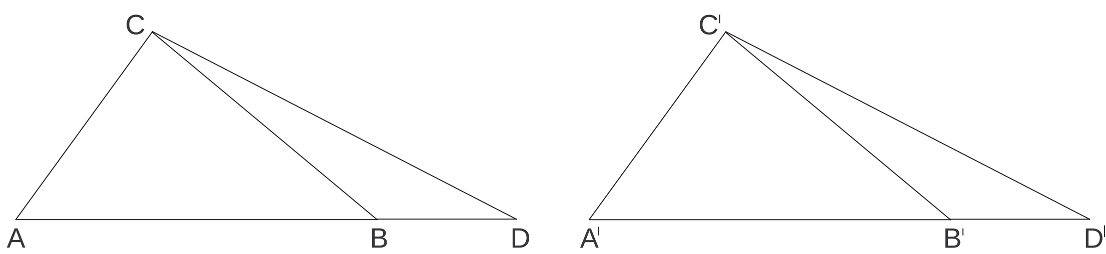


Figure 6: Supplements of congruent angles are congruent

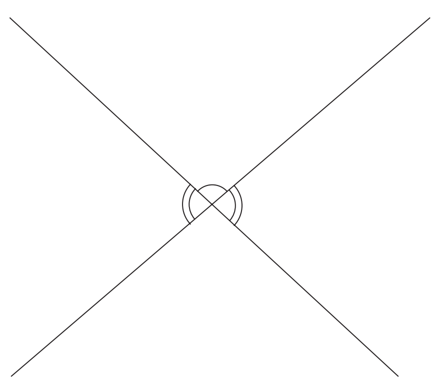

Figure 7: Vertical angles are congruent.

should supply a picture to help understand the hypotheses of the result as well as the proof.

**Lemma 1.** *Let $h$, $k$, $\ell$ be rays emanating from a point $O$. Let $h'$, $k'$, $\ell'$ be rays emanating from a point $O'$. Suppose that $h$, $k$ are on the same side (respectively, different sides) of $\ell$ and that $h'$, $k'$ are on the same side (respectively, different sides) of $\ell'$. If $\angle(h,k) \cong \angle(h',k')$ and $\angle(k,\ell) \cong \angle(k',\ell')$, then $\angle(h,\ell) \cong \angle(h',\ell')$.*

*Proof.* We complete the proof for the case where $h$, $k$ are on the same side of $\ell$ and $h'$, $k'$, the same side of $\ell'$. Using congruence of supplements of congruent angles, the reader can supply details of the other case.

Changing labels if necessary, we can suppose $h$ is interior to $\angle(k,\ell)$. Pick $K \in k$, $L \in \ell$ and choose $K' \in k'$, $L' \in \ell'$ so that $OK \cong O'K'$, $OL \cong O'L'$. Since $h$ is interior to $\angle(k,\ell)$, $KL \cap h = H$, a unique point. Choose $H' \in h'$ to satisfy $OH \cong O'H'$. We claim that $H' \in \overline{K'L'}$.

Note that we have SAS for $\Delta OKL \cong \Delta O'K'L'$ and for $\Delta OHL \cong \Delta O'H'L'$. Following through, we get $\angle OLK \cong \angle O'L'K'$ and $\angle OLH = \angle OLK \cong \angle O'L'H'$. By uniqueness of congruent angles on a given side of a ray, $\angle O'L'K' \cong \angle O'L'H'$ implying that $\overline{L'K'} = \overline{L'H'}$, i.e., $H' \in \overline{K'L'}$ as claimed. Since $KL \cong K'L'$, $HL \cong H'L'$, it follows that $KH \cong K'H'$.

Using supplements of $\angle OHL \cong \angle O'H'L'$, $KH \cong K'H'$ and $\angle OKH \cong \angle O'K'H'$, we get ASA for $\Delta OKH \cong \Delta O'K'H'$, implying that $\angle(h,k) \cong \angle(h',k')$. $\square$

Recall that we defined a right angle as one that was congruent to its supplement. We now have a procedure that allows us to construct a right angle on a given line.

**Theorem 8.** *We can construct a right angle.*

*Proof.* Fix a point $O$ and a ray $h$ emanating from $O$. Choose a point $B$ off the line $\overline{h}$ and let $k$ be the ray emanating from $O$ passing through $B$. Construct an angle congruent to $\angle(h,k)$, by taking a ray $k'$ emanating from $O$ on the side of $\overline{h}$ that does not contain $B$. Choose $B'$ on $k'$ so that $OB \cong OB'$. Let $A$ be the point where $\overline{h}$ intersects $\overline{BB'}$. Consider two cases.

Case 1: $A = O$. Here $\angle(h,k)$ and $\angle(h,k')$ are supplements and congruent so already are right angles.

Case 2: If $A \neq O$ then we have SAS for $\Delta AOB$ and $\Delta AOB'$ since $AO \cong AO$, $\angle AOB \cong \angle AOB'$ by construction, and $OB \cong OB'$, also by

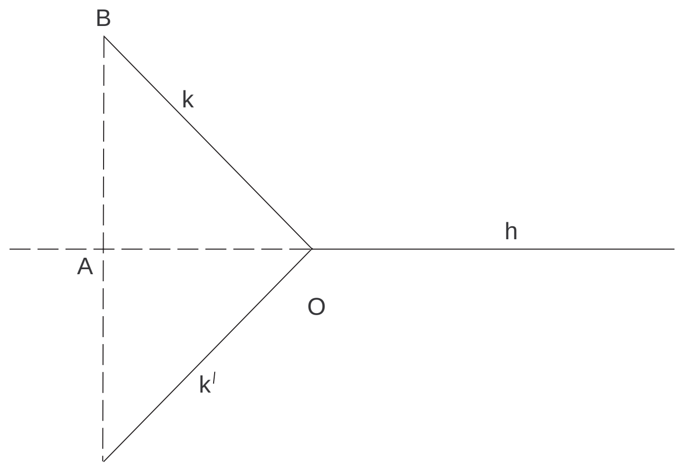


Figure 8: Construct a right angle

construction. The implication is that $\angle OAB \cong \angle OAB'$. Notice that these are supplements, thus must be right angles. $\square$

The next result brings us closer to SSS as well as transitivity for angle congruence.

**Lemma 2.** *Let $a$ be a line through points $A$ and $B$. Let $C_1$ and $C_2$ be on opposite sides of $a$ and suppose $AC_1 \cong AC_2$ and $BC_1 \cong BC_2$. Then $\angle ABC_1 \cong \angle ABC_2$.*

*Proof.* [6] Figure 9 shows one arrangement of points that satisfies the hypotheses of the lemma. Using pons asinorum, we get congruent angles as marked. We invoke Lemma 1 to subtract congruent angles so that $\angle BC_1A \cong \angle BC_2A$. Now SAS applies to $\Delta BC_1A \cong \Delta BC_2A$. As corresponding angles, $\angle ABC_1 \cong \angle ABC_2$.

The reader should complete the proof by addressing cases in which $C_1C_2$ intersects $AB$, at an endpoint or otherwise. $\square$

We must work around the fact that we do not have transitivity of angle congruence in the next proof.

[6] I am indebted to Wilson Stothers for pointing out errors in a previous version of this proof.

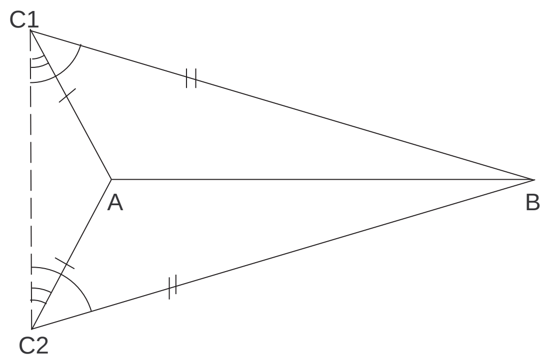


Figure 9: This is one configuration possible given the hypotheses of Lemma 2.

**Theorem 9 (SSS or the third congruence theorem for triangles).** *If in triangles $\Delta ABC$, $\Delta A'B'C'$, corresponding sides are congruent, then corresponding angles are congruent as well so that the triangles themselves are congruent.*

*Proof.* Construct one angle congruent to $\angle BAC$ at $A'$ on each side of the ray emanating from $A'$ through $C'$. $B'$ is on same side of $A'C'$ as one of the rays making these angles. On that ray, take the point $B_1$ so that $AB \cong A'B_1$. Take $B_2$ on the other ray so that $AB \cong A'B_2$. We have $CA \cong C'A'$ by assumption, $\angle BAC \cong \angle B_1A'C'$ and $\angle BAC \cong \angle B_2A'C'$ by construction, and $AB \cong A'B_1 \cong A'B_2$, also by construction. This gives us SAS for $\Delta ABC \cong \Delta A'B_1C'$ and $\Delta ABC \cong \Delta A'B_2C'$. It follows from there that $CB \cong C'B_1$ and $CB \cong C'B_2$ implying $C'B_1 \cong C'B_2$. By Lemma 1, we have $\angle C'A'B_1 \cong \angle C'A'B_2$. Lemma 1 also applies to $A'B'C'$ and $A'B_2C'$ so that $\angle C'A'B' \cong \angle C'A'B_2$. Uniqueness of angle construction implies that $B' = B_1$. This gives us $\Delta ABC \cong \Delta A'B_1C' = \Delta A'B'C'$, as desired.

Note that since segment congruence is symmetric, we can also say $\Delta A'B'C' \cong \Delta ABC$. □

The SSS criterion for triangle congruence implies that angle congruence is an equivalence relation. The first result in this direction corresponds to Axiom III.2.

**Theorem 10.** *If $\angle(h,k) \cong \angle(h'',k'')$ and $\angle(h',k') \cong \angle(h'',k'')$ then $\angle(h,k) \cong \angle(h',k')$.*

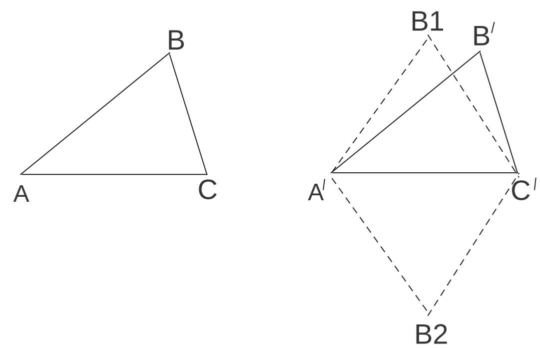


Figure 10: SSS for triangle congruence

*Proof.* Label the vertices of the angles $O$, $O'$, $O''$ respectively. On $h$, choose a point $A$. Choose $A'$ and $A''$ on $h'$, $h''$ respectively so that $OA \cong O'A' \cong O''A''$. Choose a point $B$ on $k$ and $B'$, $B''$ on $k'$, $k''$ respectively so that $OB \cong O'B' \cong O''B''$. This gives us SAS for $\Delta AOB \cong \Delta A''O''B''$ and $\Delta A'O'B' \cong \Delta A''O''B''$, thus SSS. SSS allows us to say $\Delta AOB \cong \Delta A'O'B'$, implying $\angle(h,k) \cong \angle(h',k')$ as desired. □

**Corollary 2.** *Angle congruence is an equivalence relation.*

*Proof.* Since $\angle(h',k') \cong \angle(h',k')$, $\angle(h,k) \cong \angle(h',k')$ implies by the theorem that $\angle(h',k') \cong \angle(h,k)$. Transitivity is now immediate by the theorem. □

Our next goal is to establish means for comparing angles. We need two lemmas.

**Lemma 3.** *Suppose $\angle(h,k) \cong \angle(h',k')$, with $\angle(h,k)$ emanating from vertex $O$, $\angle(h'k')$ emanating from $O'$. Let $\ell$ emanate from $O$ interior to $\angle(h,k)$. The unique ray $\ell'$ emanating from $O'$ on the $k'$ side of $h'$ that satisfies $\angle(h,\ell) \cong \angle(h',\ell')$ and $\angle(\ell,k) \cong \angle(\ell',k')$ is interior to $\angle(h,k)$.*

*Proof.* Let $\ell'$ be the unique ray emanating from $O'$ on the $k'$ side of $\overline{h'}$ with $\angle(h,\ell) \cong \angle(h',\ell')$. To show $\ell'$ is interior to $\angle(h',k')$, we need prove that if $H' \neq O'$ is a point on $h'$ and $K' \neq O'$ is a point on $k'$, then $H'K'$ intersects $\ell'$.

Let $H' \in h'$, $K' \in k'$ be different from $O'$ and choose $H \in h$ and $K \in k$ so that $OH \cong O'H'$ and $OK \cong O'K'$. This gives us SAS for $\Delta OHK \cong$

$\Delta O'H'K'$. Since $\ell$ is interior to $\angle(h,k)$, we can let $L = HK \cap \ell$. Pick $L' \in \ell$ so that $OL \cong O'L'$. This gets us SAS for $\Delta OHL \cong \Delta O'H'L'$. Note now that $\angle OHL = \angle OHK \cong \angle O'H'K'$, by the first triangle congruence and $\angle OHK = \angle OHL \cong \angle O'H'L'$ by the second. By uniqueness, $\angle O'H'K' = \angle O'H'L'$. We conclude that $L' \in H'K'$, which proves the result. $\square$

**Theorem 11.** *Let $\angle(h,k)$ and $\angle(h',\ell')$ be given, $\angle(h,k)$ emanating from $O$, $\angle(h',\ell')$ emanating from $O'$. Suppose $\ell$ is the ray emanating from $O$ on the $k$ side of $\overline{h}$ such that $\angle(h,\ell) \cong \angle(h',\ell')$. Suppose $k'$ emanates from $O'$ on the $\ell'$ side of $h'$ such that $\angle(h,k) \cong \angle(h',k')$. Then $\ell$ is interior to $\angle(h,k)$ if and only if $k'$ is exterior to $\angle(h',\ell')$.*

*Proof.* Suppose everything is as hypothesized, with $\ell$ interior to $\angle(h,k)$ but say that $k'$ is interior to $\angle(h',\ell')$. By the previous lemma, there is a unique ray $k''$ emanating from $O$, interior to $\angle(h,\ell)$ with $\angle(h,k'') \cong \angle(h',k')$. Since $\angle(h',k') \cong \angle(h,k)$, and both $k$ and $k''$ are on the $\ell$ side of $h$, that violates uniqueness of angle construction. We conclude that $\ell$ interior implies $k'$ exterior. Changing labels, we get the result in the other direction as well. $\square$

Let $\angle(h,k)$ and $\angle(h',\ell')$ be as in the theorem. We write $\angle(h,k) < \angle(h',\ell')$ provided the construction of an angle congruent to $\angle(h,k)$ using $h'$ and a ray emanating from $O'$ on the $\ell'$ side of $\overline{h'}$ yields a ray interior to $\angle(h',\ell')$. It is clear then that for any two angles $\alpha$ and $\beta$, one and only one of the following statements is true: $\alpha < \beta$, $\alpha \cong \beta$, $\alpha > \beta$.

**Theorem 12.** *All right angles are congruent.*

*Proof.* A right angle is defined as one congruent to its supplement. Let $\angle(h,k)$, emanating from $O$, and $\angle(h',k')$ emanating from $O'$, be right angles with supplements $\angle(h,\ell)$ and $\angle(h',\ell')$ respectively. Suppose $\angle(h,k) < \angle(h',k')$. Let $k''$ emanate from $O$ on the $k'$ side of $\overline{h'}$ so that $\angle(h,k) \cong \angle(h',k'')$. Note that $k''$ is interior to $\angle(h',k')$ and exterior to its supplement $\angle(h',\ell')$. This gives us

$$\angle(h,\ell) \cong \angle(h',k'') < \angle(h',k') \cong \angle(h',\ell')$$

as well as

$$\angle(h,\ell) \cong \angle(h',k'') > \angle(h',\ell')$$

which is a contradiction. $\square$

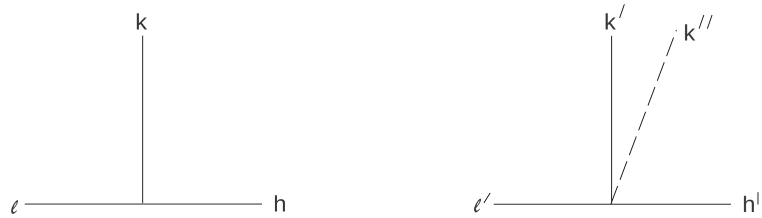


Figure 11: Right angles are congruent

**Definition 4.** *An angle greater than its supplement is* obtuse*. An angle smaller than its supplement is* acute*.*

Exterior angles play an important role in *The Elements*.

**Definition 5.** *Let $\Delta ABC$ be a triangle. The* interior angles *of the triangle are $\angle ABC$, $\angle BCA$, and $\angle BAC$. The* exterior angles *of the triangle are the supplements of the interior angles.*

**Theorem 13.** *The exterior angle of a triangle is greater than either of the interior angles that are not adjacent to it.*

*Proof.* Given $\Delta ABC$, choose $D \in \overline{AB}$ with $A * B * D*$ and $AD \cong CB$. Suppose $\angle CAD \cong \angle ACB$. Then $\Delta ACD \cong \Delta ACB$ by SAS, the implication being that $\angle ACD \cong \angle CAB$. By congruence of supplements, $\angle ACD$ would be congruent to the supplement of $\angle ACB$, implying that $D \in \overline{BC}$. Since $D \in \overline{AB}$, it follows that $D = B$ which is absurd. We conclude that $\angle CAD \ncong \angle ACB$.

Suppose next that $\angle CAD < \angle ACB$. If we construct an angle congruent to $\angle CAD$ at $C$ on the $AB$ side of $C$, we get a ray interior to $\angle ACB$ that intersects $AB$ at a point $B'$. This gives us a triangle $\Delta AB'C$, though, with exterior $\angle CAD$ congruent to interior angle $\angle ACB'$, which, as shown above, is impossible.

We conclude that $\angle CAD > \angle ACB$.

To show that $\angle CAD > \angle ABC$, use the fact that $\angle CAD$ is congruent to its vertical. □

**Exercises** Prove the following corollaries to Theorem 13.

1. In every triangle, the greater angle lies opposite the greater side. (Hint: Start by picking a point on the greater side that cuts off a segment with the length of a smaller side.)

2. A triangle with two equal angles is isosceles.

3. If $AB \cong A'B'$, $\angle BAC \cong B'A'C'$, and $\angle BCA \cong B'C'A'$, then the triangles are congruent. (This is the angle-angle-side criterion, or AAS.)

**Theorem 14.** *Every segment can be bisected.*

*Proof.* Let $AB$ be given and let $C$ be a point off $\overline{AB}$. Construct an angle congruent to $\angle CAB$ with vertex $B$, taking a ray, $k$, that extends on the side of $AB$ that does not contain $C$. Pick the point $D$ on $k$ that satisfies $AC \cong BD$.

Since $C$ and $C'$ lie on opposite sides of $\overline{AB}$, $CC'$ has a nonempty intersection with $\overline{AB}$. Call that intersection point $E$. We claim that $E$ cannot coincide with $A$ or $B$. If it did coincide with $A$, say, then $\angle BAC$ is exterior to $\Delta BAD$. But $\angle ABD$, which is interior to $\Delta BAD$, was constructed to be congruent to $\angle BAC$.

Note now that $\Delta ACE \cong \Delta BDE$ by AAS: $\angle AEC \cong BED$, since they are vertical angles; $\angle EAC \cong \angle EBD$ by construction; and $AC \cong BD$ by construction. We conclude that $AE \cong EB$, thus, that $CD$ bisects $AB$. $\square$

**Corollary 3.** *Every angle can be bisected.*

*Proof.* Let $\angle(h, k)$ emanate from $O$. Pick $A \in h$ and $B \in k$ so that $OA = OB$. Note that $\Delta AOB$ is isosceles. Let $AB$ have midpoint $C$. Triangles $\Delta ACO$ and $\Delta BCO$ have SAS by pons asinorum. It follows that $\angle AOC \cong \angle BOC$, hence, that $OC$ bisects $\angle(h, k)$. $\square$

**Axiom IV: Axiom of Parallels**

**Definition 6.** *Two lines are* parallel *provided they lie in the same plane and do not intersect.*

When we define something, it is important to establish that the definition is not vacuous, that is, that we define something that exists.

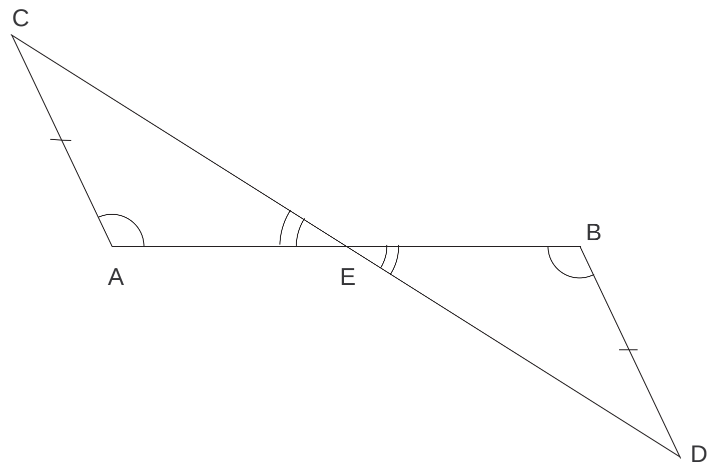


Figure 12: Segments can be bisected

Consider $a$, a line in a plane, $\alpha$. Pick two points on $a$, $A$ and $H$, and let $B$ be a point of $\alpha$ not on $a$. Let $c = \overline{AB}$. Using $B$ as the vertex and the side of $c$ containing $H$, construct the angle congruent to $\angle HAB$. Let $k$ be the constructed ray and take a point $K \in k$.

If $a$ and $\overline{k}$ had a point of intersection, say $D$, then $A$, $B$, and $D$ would form the vertices of a triangle. One of its interior angles would be $\angle HAB$ and the constructed angle at $B$ would be exterior. Since we constructed this angle congruent to $\angle HAB$, this is impossible. Thus $a$ and $\overline{k}$ are parallel.

Can there be more than one parallel to $a$ through $B$? The Parallel Postulate says no. It is important to realize that there is nothing in Hilbert's system that forces uniqueness.

**IV. (Euclid's Axiom) Let $a$ be any line and $A$ a point not on it. There there is at most one line in the plane, determined by $a$ and $A$, that passes through $A$ and does not intersect $a$.**

The construction we described above thus yields the unique parallel to $a$ through $B$.

If the line $a$ is parallel to the line $b$ write $a||b$. It is easy to see that this relation is symmetric.

**Exercise** Show that the parallel relation is transitive.

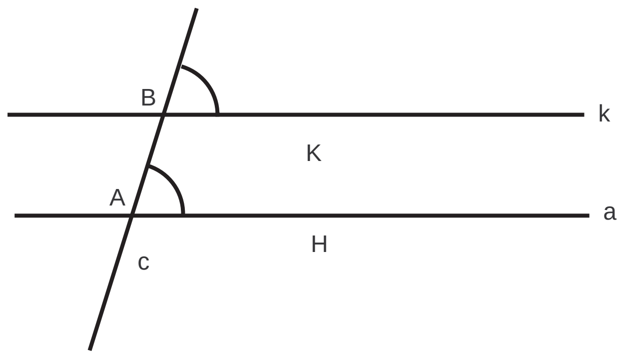


Figure 13: Construct a parallel to a given line

**Theorem 15.** *If two parallels are interesected by a third line then the corresponding and alternate angles are congruent and conversely, congruence of the corresponding or alternate angles implies the lines are parallel.*

*Proof.* Let $a$ and $b$ be parallel lines intersected at points $A$ and $B$ respectively by a third line, $c$. Let $C$ be a point on $c$. Take points $A' \in a$, $B' \in b$ on the same side of $c$. Consider the corresponding angles $\angle A'AC$ and $\angle B'BC$.

By the discussion above, we can construct a line through $B$ parallel to $A$ such that the angle at $B$ corresponding to $\angle A'AC$ is congruent to $\angle A'AC$. By uniqueness of parallels, $b$ must be that line. Thus $\angle B'BC \cong \angle A'AC$.

To get congruence of the other sets of corresponding angles and alternating angles, use supplements and verticals.

Now suppose that $a$ and $b$ are lines intersected by the line $c$ so that corresponding and alternating angles are congruent. Let $a \cap c = A$, $b \cap c = B$ and suppose $a \cap b = C$, i.e., suppose $a$ and $b$ do intersect. We have $\Delta ABC$ then with interior angle $\angle CAB$ congruent to the corresponding angle, which is exterior to $\Delta ABC$, which is absurd. $\square$

**Theorem 16.** *The angles of a triangle add up to two right angles.*

*Proof.* Let $\Delta ABC$ be given. Let $c$ be the parallel to $\overline{AB}$ through $C$. Note that we get congruent angles to $\angle A$, $\angle B$, $\angle C$ as shown, the angles congruent to $\angle A$ and $\angle B$ because of parallels, the one congruent to $\angle C$ as a vertical angle. The result follows since the constructed angles add up to a straight angle, that is, two right angles. $\square$

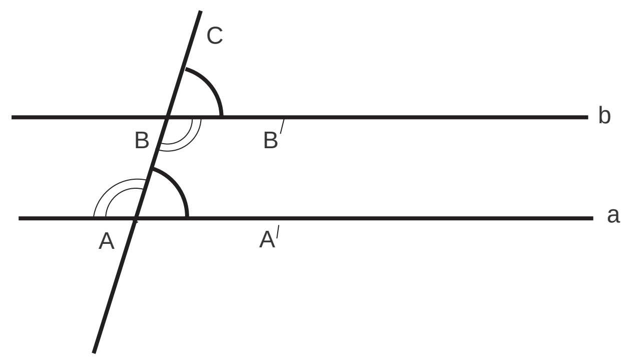


Figure 14: Alternate interior angles are congruent, as are corresponding angles.

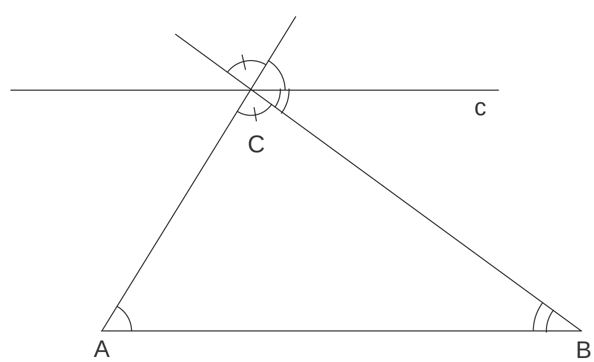


Figure 15: Angles in a triangle add up to a straight line.

It is only at this point that Hilbert introduces circles.

**Definition 7.** *If $M$ is any point in a plane $\alpha$ then the collection of all points $A$ in $\alpha$ for which the segments $MA$ are congruent to each other is called a* circle*. $M$ is the* center of the circle*.*

Besides the results we have about bisecting angles and segments, constructing parallels, etc., there are two traditional construction that Hilbert does not do explicitly that we will want in our arsenal while we pursue theorems associated to circles.

**Exercises**

1. Theorem 8 shows how to construct a right angle to a given line. Give a construction for dropping a perpendicular from a given point to a given line.

2. Give a construction for drawing a perpendicular to a line at a given point on that line.

The last set of Hilbert's axioms are to do with the geometry of the line. A thorough treatment of proportion, as in Book V of *The Elements*, calls for these sorts of ideas. We state the continuity axioms here and invoke them later, although we do not discuss the associated results in this part of the *Grundlagen*.

**V. Axioms of Continuity**

1. (Axiom of Measure or Archimedes's Axiom) If $AB$ and $CD$ are any segments then there exists a number $n$ such that $n$ segments $CD$ constructed continguously (end-to-end) from $A$, along the ray from $A$ through $B$, will pass beyond the point $B$.

2. (Axiom of Completeness) An extension of a set of points on a line with its order and congruence relations that would preserve the relations existing among the original elements as well as the fundamental properties of line order and congruence that follows from Axioms I-III and V.1 is impossible.

The Axiom of Completeness comes up when we construct the projective plane by adding points to the Euclidean plane.

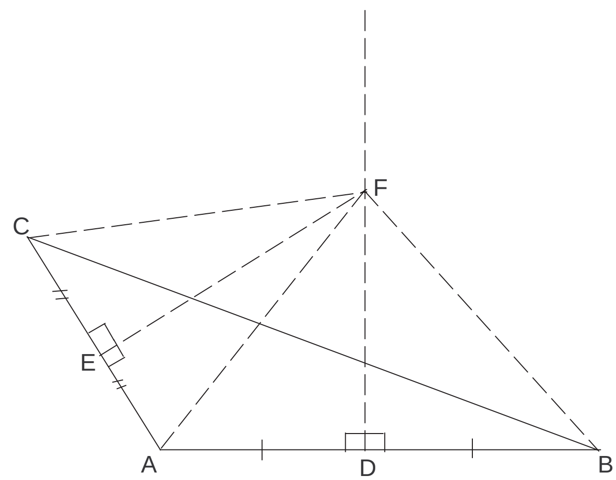


Figure 16: The circumcircle of a triangle is centered where perpendicular bisectors intersect

## 2.3 The Nine Point Circle

We use our work on Hilbert's axioms to prove some of the theorems we need to discuss the nine point circle, an interesting circle associated to a triangle. Our main sources now are [1] and [2].

There are several circles associated to a given triangle. We start our discussion by considering the *circumcircle*, the circle determined by the vertices of the triangle.

**Theorem 17.** *Three noncollinear points determine a circle.*

*Proof.* Let $A$, $B$, $C$ be noncollinear and consider $\Delta ABC$. Bisect $AB$ and $AC$ and call the midpoints of those segments $D$ and $E$ respectively. Draw the perpendicular to $AB$ at $D$, the perpendicular to $AC$ at $E$ and let $F$ be the point where those two perpendiculars intersect. (Exercise: Why must they intersect?)

We get SAS for $\Delta AFE \cong \Delta CFE$ implying that $CF \cong AF$. We also have SAS for $\Delta ADF \cong \Delta BDF$ so $AF \cong BF$. Since $A$, $B$, $C$ are equidistant from $F$, they determine the circle centered at $F$ with radius $AF$. $\square$

**Definition 8.** *A parallelogram is formed by 2 pairs of parallel lines. The*

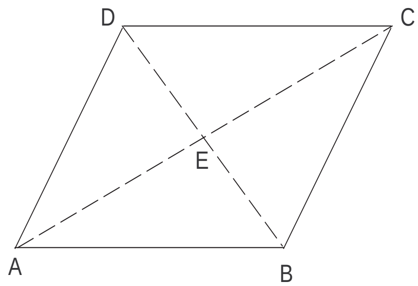


Figure 17: A parallelogram

*points where the lines intersect are the* vertices *of the parallelogram. Line segments that connect nonadjacent vertices are* diagonals*.*

Parallelograms come up in Book I of *The Elements* and prove an important tool for understanding plane figures in general. Proposition 34 from Book I describes their basic features.

**Theorem 18 (Prop. I.34).** *The opposite sides of a parallelogram are congruent. Opposite angles of a parallelogram are congruent. The diameters of a parallelogram bisect one another.*

*Proof.* Since $AC$ cuts parallels $AB$ and $CD$, alternate interior angles are congruent giving us $\angle CAB \cong \angle DCA$. Also, $AC$ cuts parallels $AD$ and $BC$ so $\angle ACB \cong \angle CAD$. Adding angles we get $\angle A \cong \angle C$. Similarly we can get $\angle B \cong \angle D$, using the diagonal $BD$.

Now $\Delta ABC \cong \Delta ADC$ by AAS, since $AC$ is a common side. This gives us $AB \cong DC$ and $AD \cong BC$.

Let $E$ be the point of intersection of $AC$ and $BD$. (Why must they intersect?) We have ASA for $\Delta AEB \cong \Delta CED$ with corresponding sides $AE \cong CE$. Similarly, $DE \cong BE$. □

**Exercise** Show that adjacent angles in a parallelogram are supplements.

**Definition 9.** *An* altitude *of a triangle is the perpendicular dropped from a vertex to the line determined by the opposite side. The* feet of a triangle *are the points on the lines determined by the sides of the triangle where*

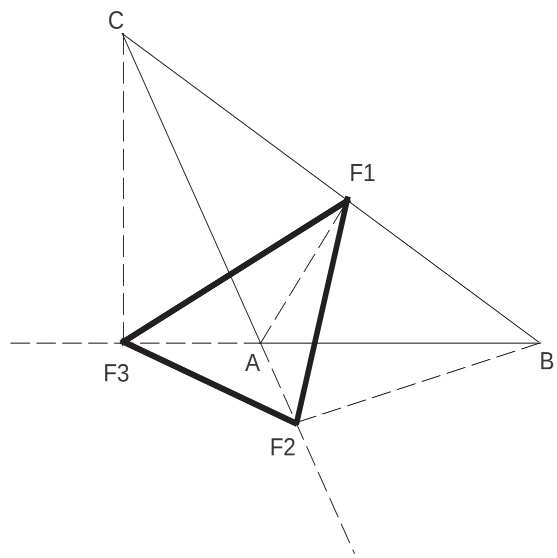


Figure 18: The orthic triangle

*they intersect the altitudes. The* orthic triangle *of* $\Delta ABC$ *is the triangle determined by the feet.*

*The* nine-point circle*,* Feuerbach circle*, or* Euler circle *of a triangle* $\Delta ABC$ *is the circumcircle of the orthic circle of* $\Delta ABC$*.*

**Definition 10.** *The line segment joining the vertex of a triangle to the midpoint of the opposite side is a* median *of the triangle.*

You may remember from your high school geometry course that the medians of a triangle intersect in a single point, that is, they are *concurrent*. To prove this, we need results about parallels and ratios, starting with a set of propositions from Book I. In this part of *The Elements*, Euclid's reference to plane figures being equal means not that they are congruent, but that they enclose equal areas. To avoid confusion, we will paraphrase Euclid, as translated by Heath. Also, when we write $\Delta ABC = \Delta A'B'C'$, it means that the areas of the two triangles are equal.

**Theorem 19 (Prop. I.35, 36, 37, 38, 39).** *Parallelograms (respectively triangles) on the same base or equal bases that lie on a line have equal areas if and only if they are in the same parallels.*

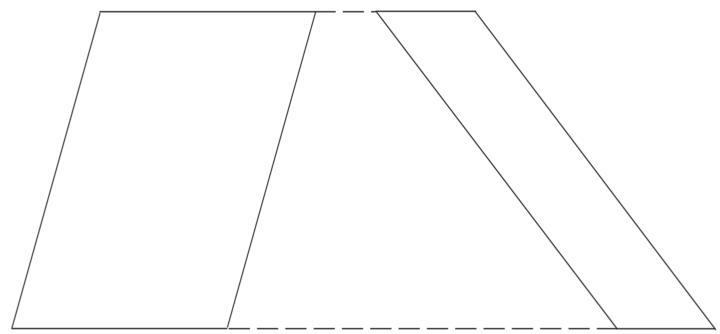

Figure 19: Rectangles in the same parallels

The notion of two parallelograms being "in the same parallels" makes sense: bases form one line and opposite sides form another. Two triangles are in the same parallels provided their bases form a line parallel to that formed by the vertices of the opposite sides. Being in the same parallels amounts to having the same height.

Euclid (Book V) and Hilbert (Axioms of Continuity) both set up the tools for discussing comparitive lengths and areas. For example, we can make sense of the idea of multiplying a line segment by 2 to get a new line segment with double the length of the original. Then we can compare the lengths of different segments. If there are positive integers $m$ and $n$ such that $m\ AB \cong n\ CD$ then $AB$ and $CD$ are *commensurate* and we write $AB/CD = m/n$. Area is trickier since it involves the notion of multiplying line segments instead of adding them but without getting too involved, we can accept the idea of comparing areas. For example, the area of a parallelogram is twice the area of a triangle formed by adjacent sides and a diagonal.

When we write $\Delta ABC/\Delta EFG$ we mean the ratio of the areas of the triangles and when we write $AB/CD$ we mean the ratio of the lengths of line segments.

We state Proposition 1 from Book VI and Proposition 39 from Book I without proof.

**Proposition 6 (Prop. VI.1).** *If triangles (respectively, parallelograms) have the same height, the ratio of their areas is equal to the ratio of their bases.*

**Proposition 7 (Prop. VI.2).** *A line parallel to one side of a triangle cuts the sides it intersects in equal proportions. Conversely, if a line cuts two*

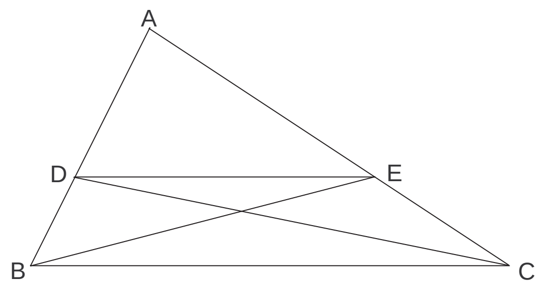


Figure 20: Parallel to the base cuts the other sides in equal proportions

*sides of a triangle in equal proportions, it is parallel to the third side of the triangle.*

*Proof.* Consider $\Delta ABC$ with $DE \| BC$, intersecting $AB$ at $D$ and $AC$ at $E$. Since $\Delta BDE$ and $\Delta CDE$ have coincident bases and are in the same parallels, $\Delta BDE = \Delta CDE$. It follows that $\Delta BDE/\Delta ADE = \Delta CDE/\Delta ADE$. Next, view $AD$ as the base of $\Delta ADE$ and $BD$ as the base of $\Delta BDE$. These two triangles have the same height as determined by the perpendicular to $AB$ from $E$. By Prop. VI.1, $\Delta BDE/\Delta ADE = BD/AD$. Similarly, $\Delta CDE/\Delta ADE = CE/EA$. It follows that $BD/AD = CE/AE$, which proves the first part of the result.

Now let the sides $AB$, $AC$ be cut so that $BD/DA = CE/EA$. We need establish that $DE \| BC$.

We have $\Delta BDE/\Delta ADE = BD/DA$. Similarly, $\Delta CDE/\Delta ADE = CE/AE$. It follows that $\Delta BDE/\Delta ADE = \Delta CDE/\Delta ADE$ so $\Delta BDE = \Delta ADE$. As $\Delta BDE$, $\Delta CDE$ have a common base, they must be in the same parallels (Prop. I.39), which proves the result. □

**Exercise** In reference to the last theorem, show that $AB/AD = AC/AE$. This is the way we usually use the theorem.

**Definition 11.** *Two triangles are* similar *provided corresponding sides are in equal proportion.*

Next is a theorem about similarity. Euclid's proof uses a construction in which the two triangles are arranged so that bases lie on a line and one vertex is shared. The argument we use is maybe a variant on that preferred

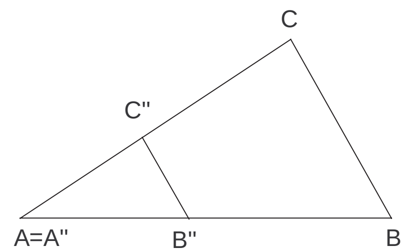


Figure 21: Equiangular triangles have corresponding sides in equal proportions

by De Morgan (see [2], p. 202.) Here we use the results we developed in Hilbert's system.

**Theorem 20 (Prop. VI.4).** *In equiangular triangles, corresponding sides are in equal proportion.*

*Proof.* Given equiangular triangles $\Delta ABC$, and $\Delta A'B'C'$ we can make a copy of $\Delta A''B''C'' \cong \Delta A'B'C'$, as follows. Let $A'' = A$, so that if we construct a copy of $\angle A'$ using $AB$ for one ray and taking the other ray on the $C$ side of $\overline{AB}$, then by uniqueness of angle construction, the second ray has to coincide with $AC$. Let $B''$ be on the $B$ side of $A''$ so that $A''B'' \cong A'B'$, and let $C''$ on $AC$ be chosen so that $A''C'' \cong A'C'$. Notice that by SAS, $\angle B'' \cong \angle B$ and $\angle C'' \cong \angle C$.

Hilbert's one theorem on parallels implies that $B''C''\|BC$. Now the exercise following Prop. VI.2 above gives us $AB/A''B'' = AC/A''C''$. We can get another pair of sides in the same ratio by constructing $\Delta A''B''C''$ starting with $B'' = B$, for example. This implies the result. $\square$

**Exercise**

1. Show that if corresponding sides of two triangles are in equal proportions, then the triangles are equiangular.

2. Show that if two pairs of sides of two triangles are in equal proportions, and the angles between are congruent, then the third pair of sides of the two triangles are in the same portion as the others.

The proof for the following we get from [1], p. 10.

**Theorem 21.** *The medians of a triangle are concurrent.*

*Proof.* Consider $\Delta ABC$ and let $D$ be the point where two medians intersect, say the ones determined by the midpoint $E$ of $AC$ and the midpoint $F$ of $AB$. Let $G$ and $H$ be the midpoints respectively of $CD$ and $BD$. Since $EF$ cuts the sides of $\Delta ABC$ into the same proportions, it is parallel to $BC$. Similarly, we see $GH \| BC$, when we consider $\Delta BCD$. This makes $EFGH$ a parallelogram. Since the diagonals of a parallelogram bisect one another, it follows that the point $D$ trisects segments $EB$, $FC$. Similarly, it must trisect the third median, which implies the result. □

There are several different "centers" associated to a triangle. The point where the medians intersect is the *centroid*, which is the center of gravity of a triangular plate of uniform density. (That was proved by Archimedes.) The center of the circumcircle is called the *circumcenter*. We did not prove this, but it is true that the circle determined by the vertices of a triangle is unique. This is another way of saying that different circles can intersect in one or two points, but no more. As a consequence, the circumcenter is unique, i.e., the perpendicular bisectors of a triangle are concurrent. The next result gives us one more center of a triangle.

**Theorem 22.** *The altitudes of a triangle are concurrent.*

The point of concurrency is the *orthocenter* of the triangle.

Before we prove the theorem, we consider the Euler line of a triangle. (The discussion and proof are from [1], p. 17.)

Let $O$ be the circumcenter and $G$ the centroid of the triangle $\Delta ABC$. (Recall that $O$ is the point where perpendicular bisectors of sides of $\Delta ABC$ intersect.) If $O = G$, then the medians of $\Delta ABC$ are perpendicular to the sides. (See Figure 22.) This gives us congruent triangles $\Delta AA'C$, $\Delta BA'A$, implying that $AB \cong AC$. Using a different median, we can repeat the argument to show that the triangle is equilateral.

If the triangle is not equilateral, the *Euler line* is $\overline{OG}$.

*Proof.* Choose $H \in \overline{OH}$ with $O * G * H$ and $OH = 3OG$, in other words, $GH = 2OG$. Recall that the segment from vertex to centroid, for example $GA$, is twice the segment from the midpoint of the opposite side to centroid, $A'G$, that is $GA = 2A'G$. Consider $\Delta OGA'$ and $\Delta AHG$.

**Exercise** Prove the following variant on Proposition 7, referring to Figure 23. Show that $OA/OA' = OB/OB'$ if and only if $AB \| A'B'$.

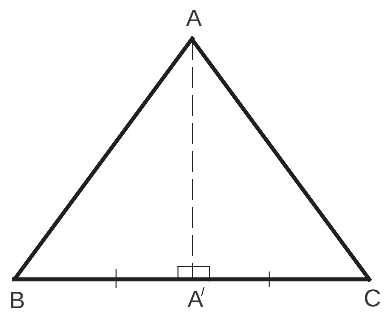


Figure 22: If medians and perpendicular bisectors coincide, the triangle is equilateral.

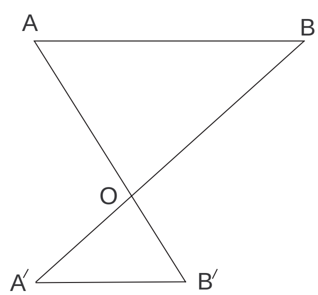


Figure 23: The triangles are similar if and only if $AB \| A'B'$.

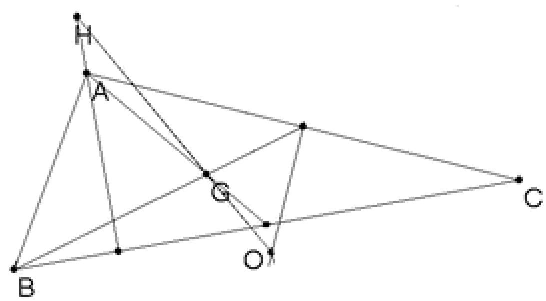


Figure 24: The Euler line passes through the centroid $G$, the circumcenter, $O$, and the orthocenter, $H$.

By the exercise, $AH \| OA'$. Moreover, $OA'$ is the perpendicular bisector of $BC$. Thus $AH$ is perpendicular to $BC$. Similarly $BH$ is perpendicular to $CA$, and $CH$ to $AB$. It follows that $H$ is the intersection of the altitudes of $\Delta ABC$. ☐

**Corollary 4.** *The circumcenter, centroid, and orthocenter of a triangle are collinear.*

We have not said much about circles in general and angles associated with them.

**Lemma 4.** *The circumcenter of a right triangle is the midpoint of its hypotenuse so the hypotenuse is a diameter of the circumcircle.*

*Proof.* Let $\Delta ABC$ be a triangle with $\angle C$ a right angle. The perpendicular bisector to $AB$ is parallel to $CB$ so intersects $AB$ at its midpoint, $C'$, by Prop. VI.2. The perpendicular bisector to $CB$ is parallel $AC$ so intersects $AB$ at $C'$ as well. It follows that $C'$ is the circumcenter for $\Delta ABC$. Moreover, the hypotenuse $AB$ must be a diameter for the circumcircle. ☐

Fix a point on a circle to be the vertex of an angle. When the rays defining the angle are determined by two other points on the circle, this is an *inscribed angle*. If the two other points are opposite ends of a diameter, the angle thus determined must be a right angle. We can see this by marking the center of the diameter and joining it to the vertex of the angle. Now use pons asinorum and supplementary angles to complete the proof. These observations together with the last lemma prove the following.

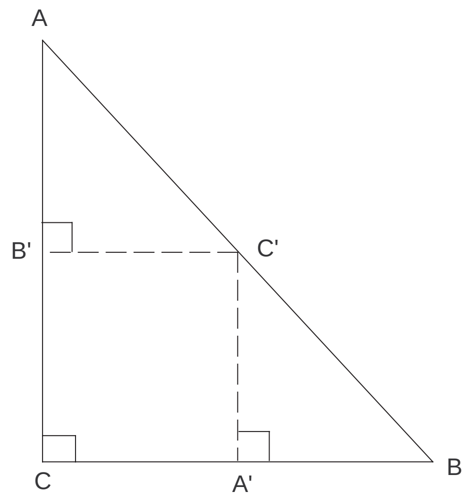


Figure 25: The circumcenter of a right triangle is the midpoint of its hypotenuse.

**Theorem 23.** *An angle inscribed in a circle is a right angle if and only if its sides emanate from points at either end of a diameter.*

This was an easy theorem to prove but is an instance of a well-known theorem that is a bit more work. We leave it as an exercise to prove in just one case of the reader's choosing.

**Exercise** Let $\alpha$ be an angle inscribed in a circle. Let $\beta$ be the associated *central angle*, i.e., the vertex of $\beta$ is the center of the circle and its sides are the radii determined by the points where the sides of $\alpha$ intersect the circle. Prove that $\beta = 2\alpha$. Choose a case to make the proof more accessible.

It is an easy step from there to see that a circle is the locus of points determined by the ends of a diameter, together with the vertices of all the right angles with sides passing through the endpoints of the diameter.

The proof of the next is from [1], p. 18.

**Theorem 24 (The Nine Point Circle).** *The midpoints of the three sides of a triangle, the midpoints of the three segments joining the vertices to the orthocenter, and the feet of the three altitudes of the triangle, all lie on a circle.*

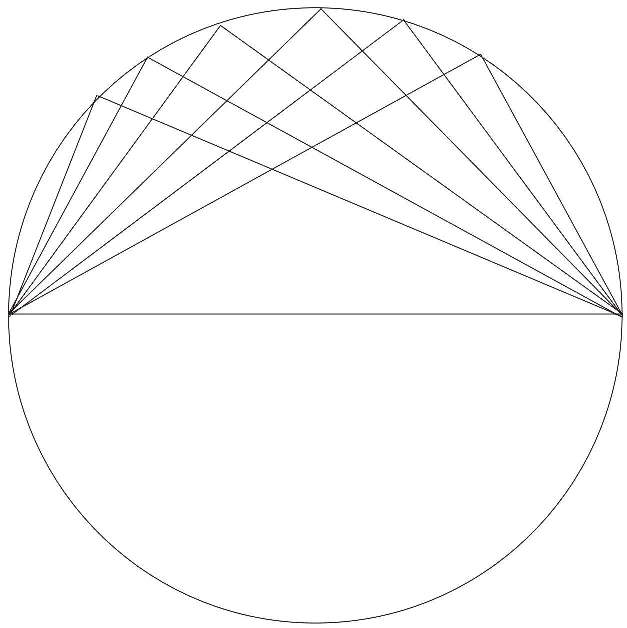

Figure 26: An inscribed angle is a right angle if and only if its sides are determined by the endpoints of a diameter.

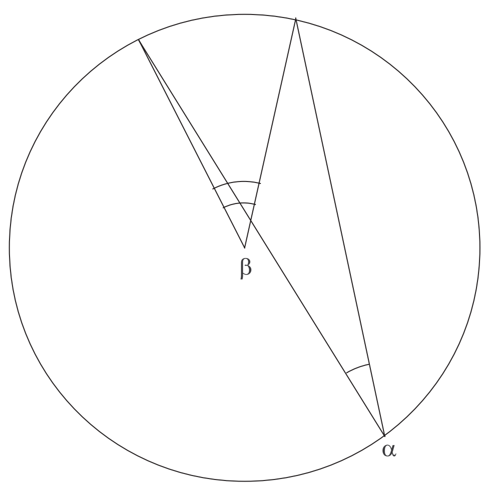


Figure 27: The inscribed angle is $\alpha$. The central angle is $\beta$.

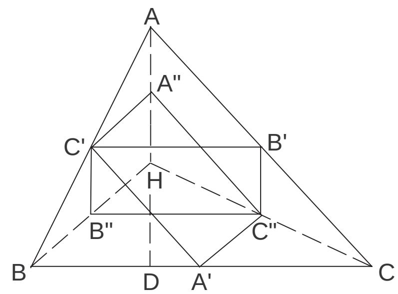


Figure 28: Proof of the nine point circle

*Proof.* Let $\Delta ABC$ be given and let $A'$ be the midpoint of $BC$, $B'$ be the midpoint of $AC$, and $C'$ be the midpoint of $AB$. Let $H$ be the orthocenter and let $A''$ be the midpoint of $AH$, $B''$ the midpoint of $BH$, and $C''$ the midpoint of $CH$. Since $C'B'$ cuts sides $AB$ and $AC$ in half, Prop. VI.2 implies $C'B' \| BC$. Next consider $\Delta BHC$ and note that since $B''$ and $C''$ cut $BH$ and $CH$ in half, $B''C'' \| BC$ as well. Considering $\Delta AHC$, we see that since $B'$ cuts $AC$ in half and $C''$ cuts $CH$ in half, $B'C'' \| AH$. Similarly, $C'B'' \| AH$. This gives us a parallelogram, $C'B'C''B''$.

Since $C'B''$ and $B'C''$ are both parallel to $AH$ which is perpendicular to $BC$, which is parallel to $B''C''$, we see that $C'B'C''B''$ is actually a rectangle.

We claim that in a rectangle, the diameters are equal. The claim follows once we notice that $\Delta C'B''C'' \cong \Delta B'C''B''$, the right triangles formed by the two different diameters of rectangle $C'B'C''B''$.

Next, we turn our attention to the quadrilateral $C'A''C''A'$. An argument similar to the one just advanced shows that this is a rectangle as well. It shares a diameter with $C'B'C''B''$, viz., $C'C''$. The three diameters $C'C''$, $A''A'$, $B'B''$, are concurrent so are diameters of a circle with center this point of concurrency.

Finally, let $D$ be the foot of the altitude from $A$ to $BC$. Notice that $\angle A''DA'$ is a right angle, so $D$ must lie on the circle with diameter $A''A'$. The same argument applies to the other three feet. This completes the proof. $\square$

## 2.4 The Projective Plane

The material here is taken mostly from [3].

The title of this subsection is a bit misleading: there are many different types of projective planes. From our point of view right now, though, the projective plane is what you get when you add *ideal points* to the Euclidean plane, as developed via using Hilbert's axioms. The effect of adding these points is to remove the special status of parallel lines. In other words, in a projective plane, any two lines intersect. The formal construction is as follows.

Allowing that a line in the Euclidean plane is parallel to itself, we make parallelism an equivalence relation. The lines in the Euclidean plane are thus partitioned into nonempty, nonintersecting equivalence classes, each class determined by a direction. All the lines in given class then have the same direction. Add a single point, called an *ideal point*, to the Euclidean plane for each equivalence class of lines. (To distinguish them from the ideal points, call the points of the original Euclidean plane *ordinary points*.) Decree that all the lines in the class contain the associated ideal point. Further decree that the collection of ideal points comprise a line, the *ideal line* or the *line at infinity*. The resulting collection of points and lines is the *real projective plane*.

**Theorem 25.** *Two distinct points determine a unique line.*

*Proof.* If the two points are ordinary points, the theorem is just a restatement of two of Hilbert's incidence axioms. If the two points are ideal, they both lie on the line at infinity. Every ordinary line in the plane contains exactly one ideal point, so the line at infinity must be the only line containing a given pair of ideal points. If one point is ordinary and the other ideal, the ideal point identifies a single parallel class, by construction. We claim that the ordinary point belongs to exactly one line in that class.

To prove the claim, pick a line out of the parallel class associated to the ideal point. If the ordinary point is on that line, we are done. If not, construct a parallel to the chosen line through the ordinary point. By Hilbert's Parallel Axiom, there is only such parallel. This proves the claim and the theorem. □

**Theorem 26.** *Two distinct lines meet in one and only one point.*

*Proof.* If the lines are both ordinary, then Hilbert's Theorem 1 says that they intersect in exactly one point or they do not intersect. If they do not intersect, they belong to the same parallel class so intersect in a single ideal point. If one line is ordinary and the other ideal, the ordinary line belongs to a single parallel class so intersects the ideal line in the unique ideal point associated to that parallel class. □

The critical nature of these two theorems is that they establish once and for all that we need not distinguish between ordinary and ideal points, nor between ordinary lines and the line at infinity. In the projective plane, all points are created equal and all lines are created equal.

This is a good time to review Hilbert's axioms with an eye towards which might apply in this setting. We start with the last set, the axioms of continuity.

**V. Axioms of Continuity**

1. (Axiom of Measure or Archimedes's Axiom) If $AB$ and $CD$ are any segments then there exists a number $n$ such that $n$ segments $CD$ constructed continguously (end-to-end) from $A$, along the ray from $A$ through $B$, will pass beyond the point $B$.

2. (Axiom of Completeness) An extension of a set of points on a line with its order and congruence relations that would preserve the relations existing among the original elements as well as the fundamental properties of line order and congruence that follows from Axioms I-III and V.1 is impossible.

The Axiom of Completeness is particularly germane. It guarantees that we cannot add a single point to a line in Euclidean space without losing fundamental properties. By our choice of ideal points, we know that the parallel axiom no longer applies. What about the others?

**I. Hilbert's Axioms of Incidence**

1. Two points determine a line.

2. The line determined by two points is unique.

3. There exist at least two points on a line. There exist at least three noncollinear points.

4. Three noncollinear points determine a plane. Every plane contains a point.

5. The plane determined by three noncollinear points is unique.

6. If two points of a line lie in a plane, then every point of that line lies in that plane.

7. Two planes with nonempty intersection have at least two points in common.

8. There are at least four noncoplanar points.

Nothing here is violated by the addition of ideal points. We are safe in assuming that Hilbert's incidence axioms apply to the projective plane. The Order Axioms are a different matter.

**II. Hilbert's Axioms of Order**

1. If $A * B * C$ then $A$, $B$, $C$ are distinct points of a line and $C * B * A$.

2. Given $A, B \in \overline{AB}$, there is $C \in \overline{AB}$ with $A * B * C$.

3. Of any three points on a line there exists no more than one that lies between the other two.

Putting these together with the axioms of continuity, we get that in a Euclidean plane, there is a one-to-one correspondence between the points on a line and the real numbers. (There is actually quite a bit more to this than we are explicating here. This is part of a course on the real line.) Then when we write $A * B * C$, it either means $x < y < z$ or $z < y < x$, where $x$, $y$, $z$ are the real numbers associated to the points $A$, $B$, $C$. If we add a single point to a line, it cannot be associated to a number. We could think of an ideal point as representing $\infty$ but for the purposes of order, we need to find a position for it on a line. Think of putting it infinitely far to the right, for example, on a line containing an ordinary point $A$. If we hang onto the order axioms, then we must be able to find $B$ so that $A * \infty * B$. This is a logistical problem that does not seem easily resolved as $B$ would presumably occupy a position farther to the right than "infinitely far." A similar problem ensues if we try planting $\infty$ infinitely far to the left on an ordinary line. If we persist, looking for another location on a line for an ideal point, the only positions left are between ordinary points. Effectively, those points are already spoken for by real numbers.

Once we accept that the order axioms have to go, we are stuck with a world that has no rays, no angles, no segments. The notion of congruence is now gone.

The reader should be warned here: there are ways of defining metric systems on the projective plane, in particular, the Fubini-Study metric is a commonly used device. But there is no natural extension of these concepts as they apply in Euclidean geometry.

**Axioms for the Real Projective Plane**

1. There exists a line.
2. Each line has as least three points.
3. There exist four points, no three of which are collinear.
4. Two distinct points determine a unique line.
5. Two distinct lines determine a unique point.
6. There is a one-to-one correspondence between the real numbers and all but one point of a line.

The last axiom is what make this the *real* projective plane. Without it, we get more general projective planes.

**Exercise**
Verify that the object in Figure 29 represents a projective plane, that is, that it satisfies the first five axioms above. Identify the points and the lines. How many of each do we have? This is called the *Fano plane*.

### 2.4.1 Duality

The most striking feature of projective geometry is the principle of duality. This can be illustrated in the projective plane by restating the axioms switching the words "point" and "line," and switching the phrases "lie on" and "meet in." When we do that we get:

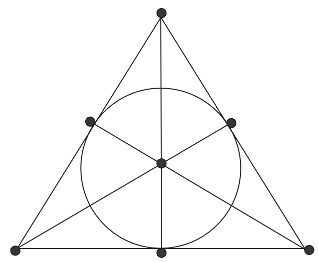

Figure 29: The Fano Plane

**Dual Axioms for the Projective Plane**

1. There exists a point.
2. Each point belongs to at least three lines.
3. There are four lines, no three of which are concurrent.
4. Two distinct lines determine a unique point.
5. Two distince points determine a unique line.
6. There is a one-to-one correspondence between the real numbers and all but one line going through a point.

Note that these could serve equally well to define the real projective plane.

**Exercises**

1. Prove that the Axioms and Dual Axioms for the Projective Plane are indeed equivalent.
2. Write the dual of each of the following statements in the projective plane.
   (a) The set of all points on a line
   (b) Four points, no three of which are collinear
   (c) Two lines, and a point on neither line

(d) The line determined by a given point and the point of intersection of two given lines

(e) All lines in the projective plane

(f) Distinct concurrent lines have only one common point.

(g) Given a line and a point not on the line, distinct lines through the given point meet the given line in distinct points.

**Definition 12.** *A* triangle *is a set of three noncollinear points (vertices) and the lines they determine (sides). A* trilateral *is a set of three nonconcurrent lines and the points they determine.*

It should be clear that triangles are self-dual, in other words, triangles and trilaterals are the same objects.

**Definition 13.** *A* complete quadrangle *is a configuration in the projective plane composed of a set of four points (vertices), no three of which are collinear, together with the six lines (sides) they determine.* Opposite sides *of a complete quadrangle are a pair of sides without a common vertex.*

The dual of a complete quadrangle is a *complete quadrilateral*.

**Exercise**
Is a complete quadrangle self-dual? Explain.

Desargues's theorem is major result of projective geometry having to do with relationships of triangles in a plane or in space. The proof of the theorem relies on embedding the projective plane in projective space. This is not a mere device to facilitate the proof of the theorem. It is, in fact, central to the validity of the theorem. In other words, there are projective planes that cannot be embedded in space and for these, Desargues's theorem is not true. These planes are called *nondesarguean*. The real projective plane is desarguean. To prove that, we need some terminology and some information about the structure of projective space.

### 2.4.2 Projective Space

We construct real projective three space, $\mathbb{R}P^3$ as follows. Start again by thinking of the lines of Euclidean three space as partitioned into equivalence classes determined by direction. Add a point to each parallel class and say that point lies on all lines in the associated parallel class and on no line from any another parallel class. As above, call the new points *ideal points* and the points from Euclidean space *ordinary points*.

The collection of ideal points in a given plane comprises a line at infinity or an *ideal line*.

An ideal point $P$ lies in a given plane $\alpha$ if and only if $\alpha$ contains a line in the parallel class associated to $P$. If $\alpha'$ is another plane parallel to $\alpha$, it contains lines in the parallel class associated to $P$ so contains $P$ as well. It follows that we are adding one collection of ideal points associated to each parallel class of ordinary planes. We decree that this collection of points comprise an ideal line. Since all the points of this ideal line lie in all the planes belonging to a given parallel class, the ideal line also lies in all the planes of the parallel class. Finally, the set of ideal points and lines together are to comprise an *ideal plane*, which contains no other points or lines.

**Theorem 27.** *Two distinct planes meet in a unique line.*

*Proof.* Suppose the planes $\alpha_1$ and $\alpha_2$ are nonparallel and ordinary. By Hilbert's system, they intersect in a unique ordinary line, $\ell$. To this, we must append the ideal point $P$ that identifies the parallel class of $\ell$. While it is clear that $\alpha_1 \cap \alpha_2$ contains no other ordinary points, maybe it contains a second ideal point, $P'$. If it does, let $\ell_1 \in \alpha_1$ and $\ell_2 \in \alpha_2$ be ordinary lines in the parallel class determined by $P'$. Since $P' \notin \ell$, neither $\ell_1$ nor $\ell_2$ is parallel to $\ell$ but $\ell_1 \| \ell_2$. Since each of $\ell_1$ and $\ell_2$ is coplanar with $\ell$, each must meet $\ell$ in an ordinary point. Say $\ell_1 \cap \ell = P_1$, and $\ell_2 \cap \ell = P_2$.

Note that $\ell_1$ and $\ell_2$ determine an ordinary plane, $\alpha$. The unique ordinary plane determined by $\ell$ and $\ell_1$ is $\alpha_1$ and the unique ordinary plane determined by $\ell$ and $\ell_2$ is $\alpha_2$ so $\alpha$ cannot contain $\ell$. It follows that the ordinary intersection $\ell \cap \alpha$ is at most one point, implying that $P_1 = P_2$, leading us to a contradiction, namely, that $\ell_1 \not\| \ell_2$. We conclude that there cannot be a second ideal point in the intersection of the two planes in this case.

Suppose $\alpha_1$ and $\alpha_2$ are ordinary and parallel. The result follows by our construction of $\mathbb{R}P^3$ in this case, as well as in the remaining case where $\alpha_1$ is ordinary and $\alpha_2$ is ideal. $\square$

**Theorem 28.** *Two distinct points determine a unique line.*

*Proof.* The proof of Theorem 25 is largely valid here. The case we must speak to is when both points are ideal since there is more than one ideal line in $\mathbb{R}P^3$.

Let $P$ and $Q$ be ideal points. By the definition of ideal points, it is clear that there are no ordinary lines containing both $P$ and $Q$. Take any ordinary point $O$ and consider the ordinary lines $\ell$, determined by $O$ and $P$, and $m$, determined by $O$ and $Q$. It is clear that $\ell$ and $m$ are distinct thus determine an ordinary plane $\alpha$. Now $P$ and $Q$ belong to the ideal line in $\alpha$. If they belonged to a second ideal line, this line would have to belong to $\alpha$ as well, contradicting the last result. □

**Theorem 29.** *Three noncollinear points determine a unique plane.*

*Proof.* Let $A$, $B$, $C$ be the three noncollinear points. If all are ideal, they determine the unique ideal plane by construction. Suppose $A$ is ordinary. Let $\ell = \overline{AB}$ and $m = \overline{AC}$. Since they contain an ordinary point, $A$, $\ell$ and $m$ are ordinary lines and they are distinct since $A$, $B$, $C$ are noncollinear by assumption. Since two ordinary lines determine a unique ordinary plane, this proves the result. □

**Theorem 30.** *Three distinct planes not containing a common line intersect in a unique point.*

*Proof.* The proof is left as an exercise. □

**Theorem 31.** *If two distinct lines meet in a point, they determine a unique plane.*

*Proof.* If both lines are ordinary, they determine a unique ordinary plane by Hilbert. If both are ideal, they determine the unique ideal plane by construction. Suppose one is ordinary and one ideal. The ideal line determines a parallel class of ordinary planes. Since the ordinary line intersects the ideal line, the ordinary line must belong to one of these parallel planes. But it can only belong to one so this is the unique plane the two lines determine. □

**Theorem 32.** *A line and a plane not containing the line meet in one and only one point.*

*Proof.* The ideal plane contains all ideal lines so line and plane cannot both be ideal. If both are ordinary, then the set of ordinary points in their intersection is either empty or a single point. If empty, then the line is parallel to a single parallel class of lines in the plane so the line and plane share the single ideal point associated to that class of parallels. If the intersection contains a single ordinary point, the given line cannot be parallel to any lines in the plane so there can be no more than the single ordinary point in the intersection of the line and plane.

If the line is ordinary and the plane ideal, the line intersects the plane in the unique ideal point associated to its parallel class. If the line is ideal and the plane ordinary, then consider the parallel class of planes associated to the line. The given plane is not in that parallel class so intersects any of the planes in it in an ordinary line. This line belongs to a unique parallel class and the associated ideal point must be the unique point of intersection of the ideal line and the ordinary plane. $\square$

**Theorem 33.** *A line and a point not on the line determine a unique plane.*

*Proof.* If both are ordinary, the result follows by Hilbert. If both are ideal, they determine the unique ideal plane. If the point $P$ is ideal and the line $\ell$ ordinary, consider the ideal line determined by $P$ and the ideal point of $\ell$. This ideal line determines a parallel class of ordinary planes exactly one of which contains $\ell$. This is the unique plane determined by $\ell$ and $P$. Now suppose $P$ is ordinary and $\ell$ ideal. There is exactly one ordinary plane in the parallel class determined by $\ell$ that contains $P$. $\square$

**Exercise** Prove that if two points lie in a plane, the line they determine lies in that plane.

Now we see that we need not distinguish between ordinary and ideal points, lines, and planes in projective 3-space.

### 2.4.3 Duality in Projective 3-Space

The duality principle applies to all projective spaces, not just real projective spaces, that is, those based on $\mathbb{R}$, like $\mathbb{R}P^2$ and $\mathbb{R}P^3$. In projective 3-space, *points* and *planes* are dual, while lines are self-dual. If the *dimension* of a point is 0, of a line is 1, a plane 2, etc. then duality in projective $n$-space allows us to switch points with $n-1$-dimensional objects, lines with $n-2$-dimensional objects, etc. In other words, the switch is allowed between objects when the sum of their dimensions is one less than the ambient space.

**Exercise**

1. Arrange the theorems in the previous subsection in dual pairs.

2. Write the dual for each of the following, as statements or objects in projective 3-space.

   (a) the set of points on a line
   (b) the set of points on a plane
   (c) the set of planes containing a given line
   (d) the set of lines passing through a common point
   (e) the set of lines lying on a given plane
   (f) the set of planes in space
   (g) the set of lines in space
   (h) the set of all points in space

### 2.4.4 The Theorem of Desargues

**Definition 14.** *Two triangles $\Delta ABC$, $\Delta A'B'C'$ are* perspective from a point *$O$ provided $\overline{AA'}$, $\overline{BB'}$, $\overline{CC'}$ are concurrent at $O$. In this case, $O$ is a* center of perspectivity. *The triangles are* perspective from a line *$\ell$ provided the points $\overline{AB} \cap \overline{A'B'} = P_1$, $\overline{AC} \cap \overline{A'C'} = P_2$, and $\overline{BC} \cap \overline{B'C'} = P_3$ are collinear. In this case, $\ell$ is an* axis of perspectivity.

**Theorem 34 (Desargues).** *In projective 3-space, two triangles are perspective from a point if and only if they are perspective from a line.*

*Proof.* For the first part of the proof, we assume that the two triangles are not coplanar. Say $\Delta ABC$ determines a plane $\alpha$ and $\Delta A'B'C'$ determines $\alpha'$.

If $\Delta ABC$ and $\Delta A'B'C'$ are perspective from a point $O$ since $\overline{AA'}$ and $\overline{BB'}$ intersect, they determine a plane which contains $\overline{AB}$ and $\overline{A'B'}$ which thus intersect in a point $P_1$. Note that $P_1 \in \alpha \cap \alpha'$. Similarly, $\overline{AA'}$ and $\overline{CC'}$ determine a plane so that $\overline{AC} \cap \overline{A'C'} = P_2$ with $P_2 \in \alpha \cap \alpha'$. Finally, $\overline{BC} \cap \overline{B'C'} = P_3 \in \alpha \cap \alpha'$. It follows that $P_1$, $P_2$, $P_3$ lie on the line $\alpha \cap \alpha'$.

Next suppose $\Delta ABC$ and $\Delta A'B'C'$ are perspective from a line. As intersecting lines, $\overline{AB}$ and $\overline{A'B'}$ determine a plane $\pi$ in which $\overline{AA'}$ and $\overline{BB'}$ intersect. Similarly, $\overline{AA'}$ and $\overline{CC'}$ intersect in a plane $\pi'$; and $\overline{BB'}$ and $\overline{CC'}$ intersect in a third plane $\pi''$. Since $A$, $B$, $C$ are noncollinear, the planes $\pi$, $\pi'$, $\pi''$ intersect in a point, $O$. Now $\overline{AA'} = \pi \cap \pi'$, $\overline{BB'} = \pi \cap \pi''$, and $\overline{CC'} = \pi' \cap \pi''$. It follows that the three lines $\overline{AA'}$, $\overline{BB'}$, $\overline{CC'}$ are concurrent at $O$.

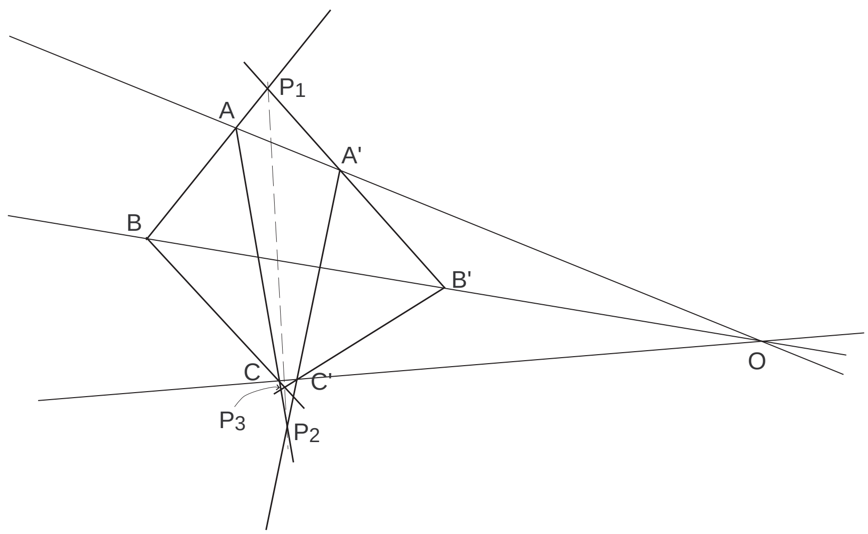


Figure 30: The theorem of Desargues is true in Euclidean space as long as all the intersections occur.

For the second part of the theorem, assume that $\Delta ABC$ and $\Delta A'B'C'$ lie in a plane $\pi$. Suppose the two triangles are perspective from a point $O \in \pi$. Let $\ell$ be a line through $O$, not lying in $\pi$. Choose $P$, $P'$ distinct points on $\ell$, both different from $O$. The lines $\overline{PA}$ and $\overline{P'A'}$ must intersect in a point $A''$ since they lie in the plane determined by intersecting lines $\overline{AA'}$ and $\overline{PP'}$. Produce points $B''$, $C''$ in a similar manner. Notice that the points $A''$, $B''$, $C''$ cannot be collinear: if they were, the planes determined by $P$ and each of the three sides of the triangle $\Delta ABC$ would coincide, implying that $A$, $B$, $C$ were collinear. We thus have a new triangle $\Delta A''B''C''$. By construction, $\Delta ABC$ and $\Delta A''B''C''$ are perspective from $P$, while $\Delta A'B'C'$ and $\Delta A''B''C''$ are perspective from $P'$. The line of perspectivity in both cases is the intersection of $\pi$ with the plane of $\Delta A''B''C$". This gives us that $\overline{A''B''} \cap \overline{AB}$ and $\overline{A''B''} \cap \overline{A'B'}$ fall on this line of intersection. Since $\overline{A''B"}$ meets this line just once, $\overline{AB}$ and $\overline{A'B'}$ intersect there as desired. The other sides of the two triangles meet on this line as well, for the same reasons.

We leave the last part of the proof to the reader. $\square$

**Exercise** Figure 30 shows what is called a *Desarguean configuration*, an arrangement of ten points. Show that any one of the ten points can serve as the center of perspectivity for a Desarguean configuration with the same ten points.

## 3 Affine Planes

The most familiar algebraic structure is possibly the *vector space*, in particular, $\mathbb{R}^n$. In this section, our goal is to understand how we can use vector spaces to model Euclidean and projective spaces. The first thing is to establish what points, lines, and planes should be and then, that they satisfy Hilbert's axioms.

We take vectors to play the role of points. Playing the role of lines, we have cosets of one dimensional spaces.

**Definition 15.** *Let $V$ be a vector space, $W$ a subspace of $V$. Let $u \in V$. The $W$-coset determined by $u$ is $u + W = \{u + w \mid w \in W\}$.*

Denote the span of a set of vectors $\{v_1, \ldots , v_n\}$ by $< v_1, \ldots, v_n >$.

The results that follow are from [5], starting with Lemma 3.2.1, p. 101.

**Lemma 5.** *For any vector space $V$ with subspaces $W, W'$, and vectors $u, u' \in V$,*

1. *$u + W = u' + W$ if and only if $u - u' \in W$;*
2. *$u + W \cap u' + W$ is empty or $u + W = u' + W$;*
3. *if $\dim V = 2$ and $\dim W = \dim W' = 1$, $u + W \cap u' + W' =$ if and only if $W = W'$.*

*Proof.*

1. $u+W = u'+W$ if and only if for every $w_1 \in W$, there is $w_2 \in W$ with $u + w_1 = u' + w_2$ which is true if and only if $u - u' \in W$.
2. If $v \in u + W \cap u' + W$, then there are $w_1, w_2 \in W$ with $v = u + w_1 = u' + w_2$, which is true only if $u - u' \in W$, i.e., $u + W = u' + W$.
3. If $W =< w >$ and $W' =< w' >$, then $W \neq W'$ means precisely that $\{w, w'\}$ is a basis for $V$. In this case, suppose $u = \alpha w + \beta w'$ and $u' = \alpha' w+\beta' w'$. Take $v = \alpha' w+\beta w'$ and notice that $u-v = (\alpha-\alpha')w \in W$, and $u' - v = (\beta' - \beta)w' \in W'$. It follows that $v \in u + W \cap u' + W'$.

∎

**Lemma 6.** *Two distinct vectors $v$, $v'$ determine a unique* 1*-dimensional coset in* $V$.

*Proof.* Let $W =< v - v' >$. Since $v - v' \in W$, $v' + (v - v') = v \in v' + W$ so by Lemma 5.1, $v + W = v' + W$. Suppose $u + W'$ is another one dimensional coset containing $v$ and $v'$. The again by Lemma 5.1, $v + W' = v' + W' = u + W'$, which implies that $v - v' \in W'$, thus that $W' =< v - v' >= W$. □

**Lemma 7.** *Two one dimensional cosets intersect in at most one point.*

*Proof.* Suppose $v, v' \in u + W \cap u' + W$. Then $v = u + w_1$, some $w_1 \in W$ and $v' = u + w_2$, some $w_2 \in W$ so that $v - v' \in W$. Similarly, $v - v' \in W'$ implying either $v = v'$ or $< v - v' >= W = W'$. □

**Definition 16.** *A* field *is a set $F$ with at least two distinct elements,* 0 *and* 1*, together with two commutative and associative binary operations $\oplus$ and $\odot$ such that the following axioms are satisfied.*

1. 0 *is an identity for $\oplus$, i.e., $0 \oplus a = a$ for all $a \in F$.*
2. *For every $a \in F$, there is $a' \in F$ such that $a \oplus a' = 0$.*
3. 1 *is an identity for $\odot$, i.e., $1 \odot a = a$ for all $a \in F$.*
4. *Every element $a \in F$, except* 0*, there is $\hat{a} \in F$ such that $a \odot \hat{a} = 1$.*
5. *For all $a, b, c \in F$, $a \odot (b \oplus c) = (a \odot b) \oplus (a \odot c)$.*

**Example 1.** $\mathbb{R}$ *is an example of an* ordered field *because aside from satisfying the axioms for a field, it has an ordering that respects the field operations of addition and multiplication. In particular, $a < b$ implies that $a + c < b + c$ and if $0 < a$, then $b < c$ implies that $ab < ac$.*

Our favorite examples of fields besides $\mathbb{R}$ are $\mathbb{Q}$, $\mathbb{C}$ and the integers modulo $p$, as in the next example.

**Example 2.** *Let $\mathbb{Z}_p = \{\bar{0}, \bar{1}, \ldots, \overline{p-1}\}$ designate the set of residue classes of integers mod $p$. That is, $\bar{0} = \{kp \mid k \in \mathbb{Z}\}$, $\bar{1} = \{1 + kp \mid k \in \mathbb{Z}\}$, $\bar{2} = \{2 + kp \mid k \in \mathbb{Z}\}$, etc. Let $\bar{a} \oplus \bar{b} := \overline{a+b}$ and let $\bar{a} \odot \bar{b} = \overline{ab}$. We leave it as an exercise to verify that $\oplus$ and $\odot$ are well-defined and that they satisfy the axioms for a field.*

Now we are free to consider vector spaces with arbitrary fields of scalars. Unless noted to the contrary, we assume $V$ is a vector space defined over some field $F$.

Consider that if $u+W$ is a one dimensional coset in $V$ and $v \in u+W$, then $v-u \in W$ implies that $W =< v-u >$. It follows that for any $v' \in u+W$, there is $\alpha \in F$ such that $v' = u+\alpha(v-u) = (1-\alpha)u+\alpha v$. In other words, we can coordinatize the points on the line $\ell = u+W$ by $(\alpha, \beta)$, where $\alpha, \beta \in F$ satisfy $\alpha + \beta = 1$. These $\alpha$ and $\beta$ are then the *affine line coordinates* for $v'$ with respect to $v$ and $u$. On the other hand, for any $\alpha \in F$, there is a point $(1-\alpha)u + \alpha v \in u+W$: the set of points on a line in $V$ has the same cardinality as $F$. In particular, since $F$ must have at least two elements, every line in $V$ must have at least two distinct points.

We have proved the following.

**Theorem 35.** *Let $F$ be a field with at least three elements. Let $V$ be a two dimensional vector space over $F$. The collection of vectors and one dimensional cosets of $V$ form the points and lines of* the affine plane $\Pi(F)$, *which satisfies Hilbert's axioms of incidence for a plane.*

In the finite fields we have seen, there is $k$ such that

$$\overbrace{1 \oplus \ldots \oplus 1}^{k \text{ times}} = 0.$$

This also happens for certain infinite fields, the study of which extends beyond the scope of this course. When there is one such $k$, there must be a minimal value of $k$ for which it is true. That minimal $k$ is called the *characteristic of $F$.* When there is no such $k$, the field is said to have *characteristic* 0. $\mathbb{R}$, $\mathbb{C}$, and $\mathbb{Q}$ all have characteristic zero. $\mathbb{Z}_p$ has characteristic $p$. It is not difficult to verify that the characteristic of a field is either zero or a prime number.

Assume now that the characteristic of $F$ is not two. Then in $F$ we have $1 \oplus 1 = 2 \neq 0$ so that 2 has an inverse with respect to $\odot$, multiplication. For convenience, designate this inverse $1/2$. Let $u, v \in V = F^2$. Then we can define the *midpoint* of $u$ and $v$ to be $(1/2)(u+v)$.

We have $u, v \in \ell$ means $\ell = u+ < v-u >$ so that $\alpha u + \beta v \in \ell$ for all $\alpha, \beta \in F$ with $\alpha \oplus \beta = 1$. Does $1/2 \oplus 1/2 = 1$ in any field $F$ with characteristic different from 2? To see that the answer is yes, we invoke the distributive law to get

$$(1/2 \oplus 1/2) = 1/2 \odot (1 \oplus 1) = 1/2 \odot 2 = 1.$$

Next, we want to verify that in $\Pi(\mathbb{R})$, $(1/2)(u+v)$ is the same distance from $u$ as from $v$. Writing $u = (u_1, u_2)$, $v = (v_1, v_2)$ we have $(1/2)(u+v) = ((u_1+v_1)/2, (u_2+v_2)/2)$. Using the distance formula to find the distance from $u$ to $(1/2)(u+v)$, we get

$$d(u,(1/2)(u+v)) = \sqrt{(1/2(u_1+v_1)-u_1)^2 + (1/2(u_2+v_2)-u_2)^2}$$

$$= \sqrt{(1/2(v_1-u_1))^2 + (1/2(v_2-u_2))^2} = 1/2\ d(u,v).$$

By symmetry, the distance is the same from $u$ to $(1/2)(u+v)$.

**Definition 17.** *A* triangle *in* $\Pi(F)$ *is a set of* 3 *noncollinear points and the lines they determine.*

**Lemma 8.** $\{u,v,w\} \subset \Pi(F)$ *is a triangle if and only if* $\{v-u, w-u\}$ *is a basis for* $F^2$*.*

*Proof.* We leave the proof as an exercise. ☐

Lemma8 shows that if we fix any triangle in $\Pi(F)$, $\{u,v,w\}$, and an ordering of its vertices, then for each $z \in \Pi(F)$, there are unique $\lambda$, $\mu \in F$ with

$$z - u = \lambda(v-u) + \mu(w-u).$$

This gives us $z = (1-(\lambda+\mu))u + \lambda v + \mu w$. It follows that every point of $\Pi(F)$ has a unique triple of scalars $(\alpha, \beta, \gamma)$ where $z = \alpha u + \beta v + \gamma w$, where $\alpha + \beta + \gamma = 1$. These are the *affine coordinates* of $z$ with respect to $\{u,v,w\}$.

**Exercise**
Check that when $\Pi(F) = \mathbb{R}^2$ and $\{u,v,w\} = \{(0,0),(1,0),(0,1)\}$, the affine coordinates of a point $(\lambda,\mu)$ are $(1-(\lambda+\mu),\lambda,\mu)$.

We reconsider Theorem 21 in $\Pi(F)$.

**Exercise**
Let $\{u,v,w\}$ be a triangle in $\Pi(F)$. A *median* of the triangle is a line determined by one vertex and the midpoint of the line determining the opposite side. Verify that the medians of the triangle are

$$m_1 = u + < u - (1/2)v - (1/2)w >$$

$$m_2 = v + < v - (1/2)w - (1/2)u >$$

$$m_3 = w + < w - (1/2)u - (1/2)v > .$$

Now suppose the characteristic of $F$ is 3. To make it easier to think about, say $F = \mathbb{Z}_3$. In $F$ we have $2 \oplus 1 = 0$ so $-1 = 2$ and $1 = -2$. Referring to the exercise above, let $\{u, v, w\}$ be a triangle in $\Pi(F)$ with medians $m_1$, $m_2$, $m_3$. Letting $z = u + v + w$, we can rewrite $m_1 = u + < z >$, $m_2 = v + < z >$, $m_3 = w + < z >$. It follows that the medians are either identical or parallel. We claim they are parallel.

Note that $z = u + v + w = -2u + v + w = (v - u) + (w - u)$. If $m_1 = m_2$, there is $\alpha \in F$ with $u - v = \alpha(v - u) + \alpha(w - u)$. Note that $u - v = 2(v - u) + 0(w - u)$. Since $\{v - u, w - w\}$ is linearly independent, $\alpha$ is unique, implying that $\alpha = 2$ and $\alpha = 0$, impossible as characteristic of $F$ is 3. It follows that the medians are distinct and parallel.

**Theorem 36.** *Let $F$ be a field with characteristic not 2 or 3. Then the medians of a triangle in $\Pi(F)$ are concurrent.*

*Proof.* Let $\{u, v, w\}$ be a triangle in $\Pi(F)$. Let $z \in m_1$ and say $(\alpha, \beta, \gamma)$ are the unique affine coordinates determined by our triangle with vertices in the order given. We also have affine line coordinates for $z$ determined by $m_1$: $z = u + \delta u - (1/2)\delta v - (1/2)\delta w$. Note that $z = (1+\delta)u - (1/2)\delta v - (1/2)\delta w$. Since $1 + \delta - (1/2)\delta - (1/2)\delta = 1$, it follows by uniqueness of affine coordinates that $\alpha = 1 + \delta$, $\beta = (-1/2)\delta$, $\gamma = (-1/2)\delta$. Similarly, $z$ belongs to $m_2$ if and only if there is $\xi \in F$ with $\alpha = (-1/2)\xi$, $\beta = 1 + \xi$, $\gamma = (-1/2)\xi$. We conclude that $m_1$ and $m_2$ intersect if and only if the following equations can be solved simultaneously.

$$\begin{array}{ccccccc} & & (-1/2)\xi & = & 1 & + & \delta \\ 1 & + & \xi & = & & & (-1/2)\delta \\ & & (-1/2)\xi & = & & & (-1/2)\delta \end{array}$$

By the last equation, $\xi = \delta$. From the first equation $\xi = -2/3 = \delta$. The affine coordinates of the unique point of intersection of $m_1$ and $m_2$ is then $(1/3, 1/3, 1/3)$. We leave the rest of the proof to the reader. $\square$

**Exercises**

1. Complete the proof of the last theorem by verifying that the point with affine coordinates $(1/3, 1/3, 1/3)$ lies on $m_3$.

2. Consider the triangle $\{(0,0),(1,0),(0,1)\} \subset \Pi(\mathbb{Z}_5)$. What is the point of concurrency of its medians?

3. Let $\{u, v, w\}$ be a triangle in $\Pi(F)$. Let $x$, $y$, $z$ be points in $\Pi(F)$ with affine coordinates given respectively by $(\alpha_1, \beta_1, \gamma_1)$, $(\alpha_2, \beta_2, \gamma_2)$, $(\alpha_3, \beta_3, \gamma_3)$. Show that $x$, $y$, $z$ are collinear if and only if

$$\det \begin{pmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \\ \alpha_3 & \beta_3 & \gamma_3 \end{pmatrix} = 0.$$

## 3.1 The Affine Group

**Definition 18.** *Let $G$ be a set of mappings on a set $S$. Suppose $e \in G$ is the identity mapping, that is, $e(s) = s$ for all $s \in S$. If $G$ is closed under function composition, and if every element in $G$ has an inverse in $G$ then $G$ is a* group.

The most fundamental example of a group is the set of permutations on any set. The best way to understand this is to write out all the permutations on a set of three objects $\{a, b, c\}$ and see what happens if you start composing them. In some sense, all groups are groups of permutations. This is the content of *Cayley's Theorem*, one of the big theorems you learn in a beginning course in abstract algebra.

Groups are a critical element in the modern study of geometry.

**Example 3.** *The set of invertible linear transformations on any vector space, $V$, is a group designated $GL(V)$. This is easy to check and the reader is urged to do so. When $\dim_F V = n$, $GL(V)$ can be identified with the set of nonsingular $n \times n$ matrices with entries in $F$. In this case we often write $GL(V) = GL_n(F)$.*

**Example 4.** *A vector space $V$ is a group which we also designate $V$. In what sense is a vector space a collection of mappings on a set? For any $v \in V$, define $t_v : V \to V$ by $t_v(u) = v + u$, for all $u \in V$. We then have*

*$t_v \circ t_w(u) = t_v(t_w(u)) = t_v(w+u) = v+w+u = t_{v+w}(u)$ so that $t_v \circ t_w = t_{v+w}$. The identity mapping is $t_\theta$, where $\theta$ is the zero vector. The inverse of $t_v$ is $t_{-v}$. Notice that unlike $GL(V)$, $V$ is a commutative group, that is, while function composition is generally not commutative, $t_v \circ t_u = t_{v+u} = t_{u+v} = t_u \circ t_v$. Any group in which the operation is commutative like this is called an* abelian group.

**Definition 19.** *Let $V = F^2$. Fix $T \in GL(V)$ and $v \in V$. Define $A(T,v) : V \to V$ by $A(T,v)(u) = T(u) + v$, for all $u \in V$. The* affine group $\mathrm{Aff}(V)$ *is the set $\{A(T,v) | \; T \in GL(V), v \in V\}$.*

As a set, $\mathrm{Aff}(V)$ is the Cartesian product of $GL(V)$ and $V$, in other words, an element of $\mathrm{Aff}(V)$ can be identified with an ordered pair $(T,v)$, $T \in GL(V)$, $v \in V$. There are actually many different ways to define "products" of groups. The affine group is an example of what is called a *semi-direct product* of $GL(V)$ and $V$.

**Lemma 9.** *The affine group maps lines to lines in $\Pi(F)$.*

*Proof.* Let $u + \alpha w \in \ell = u + < w >$. For any $T \in GL(V)$, $v \in V$, we have $A(T,v)(u+\alpha w) = T(u+\alpha w)+v = T(u)+\alpha T(w)+v \in v+T(u)+ < T(w) >$. Since $T$ is an invertible linear transformation, $T(w)$ cannot be zero if $w \neq \theta$, thus the image of the line $\ell$ under $A(T,v)$ is a line $v + T(\ell)$. ☐

It is clear that if $x \in \ell \cap \ell'$, where $\ell$ and $\ell'$ are lines, then for any affine transformation $A(T,v)$, $A(T,v)(x) \in A(T,v)(\ell) \cap A(T,v)(\ell')$.

**Lemma 10.** *The affine group maps triangles to triangles in $\Pi(F)$.*

*Proof.* By Lemma 8, $\{x,y,z\}$ is a triangle in $\Pi(F)$ if and only if $\{y-x, z-x\}$ is a basis for $V$. Let $A(T,v) \in \mathrm{Aff}(V)$. We have $A(T,v)\{x,y,z\} = \{T(x) + v, T(y)+v, T(z)+v\}$, and since $T \in GL(V)$,

$$\{T(y-x), T(z-x)\} = \{T(y) - T(x), T(z) - T(x)\} = \{T(y) + v - (T(x)+v), T(z) + v - (T(x) +$$

is also a basis for $V$. Lemma 8 implies that $A(T,v)\{x,y,z\}$ is a triangle. ☐

**Theorem 37.** *Let $\ell$ and $\ell'$ be lines in $\Pi(F)$. There is an affine transformation $A(T,v)$ such that $A(T,v)(\ell) = \ell'$. If $\{x,y,z\}$ and $\{x',y',z'\}$ are triangles, there is a unique $A(T,v) \in \mathrm{Aff}(V)$ with $A(T,v)(\{x,y,z\}) = \{x',y',z'\}$.*

*Proof.* Let $\ell = x + < y >$ and $\ell' = x' + < y' >$ be distinct lines in $\Pi(F)$. Choose $T \in GL(V)$ so $T(y) = y'$. Let $v = x' - T(x)$. Then $A(T, v)(x + < y >) = T(x) + v + < T(y) > = x' + < y' >$.

Given triangles $\{x, y, z\}$ and $\{x', y', z'\}$, there is a unique element $T \in GL(V)$ that maps $\{y - x, z - x\}$ to $\{y' - x', z' - x'\}$. Consider that

$$A(T, -T(x) + x')\{x, y, z\} = \{x', T(y) - T(x) + x', T(z) - T(x) + x'\} = \{x', y' - x' + x', z' - x' + x$$

If $A(T, v)(x) = T(x) + v = x'$, then $v$ is uniquely determined. The result follows. $\square$

Going back to our definition of "group," we see it is easy to verify that if $G$ is a group of mappings on a set $S$, and if $S' \subset S$, then the set $H \subset G$ of elements in $G$ that map $S' \to S'$ forms a group inside $G$. Thus, $H$ is a *subgroup* of $G$. Elements of $H$ are said to leave $S'$ *invariant.*

Let $H$ be the subgroup of $\mathrm{Aff}(V)$ that leaves a given triangle invariant, that is, that maps the triangle to itself with a permutation of its vertices. The collection of $H$ cosets is $\{g \circ h | g \in \mathrm{Aff}(V), h \in H\}$. These can actually serve as triangles. This is typical of the Erlangen Program of Felix Klein, a hugely influential idea described in a lecture of Klein's dating to 1872. At that point in history, geometry was in a state of disarray. The notion that groups could and should serve as a unifying force in geometry turned out to be profound. It remains at the heart of a great deal of mathematics being discovered today.

**Exercises**

Define a *complete quadrilateral* in $\Pi(F)$ to be a set of four points, no three of which are collinear, and the lines they determine. The four points are the *vertices* of the quadrilateral. A quadrilateral $\{u, v, w, x\}$ is a *parallelogram* provided its vertices can be relabeled if necessary so that
$u + < v - u > \| x + < w - x >$ and $u + < x - u > \| v + < w - v >$. The lines $u + < w - u >$ and $v + < x - v >$ are the *diagonals.*

1. Prove that a quadrilateral $\{u, v, w, x\}$ is a parallelogram if and only if one of its vertices is the sum of the other two minus the third.

2. Let $\{u, v, w, x\}$ be a parallelogram in $\Pi(F)$, where the characteristic of $F$ is not two. Show that the diagonals of the parallelogram bisect one another.

3. Prove that in an affine plane over a field of characteristic not two, every parallelogram is the image under some affine transformation of $\{(0,0),(2,0),(2,2),(0,2)\}$.

4. Consider $\Pi(\mathbb{C})$, the affine plane over $\mathbb{C}$. Let $\varphi(z_1, z_2) = (\bar{z}_1, \bar{z}_2)$, where $\bar{z}$ is the complex conjugate. (See below for more details, if this is unfamiliar.) Prove that $\varphi$ sends lines to lines and triangles to triangles but that it is not an affine transformation.

# 4 The Quaternions

This section represents something of a digression. Our objective is to describe how the quaternions define rotations in $\mathbb{R}^3$. By the end of the section, we should see that it is not as much of a digression as it first appears. The quaternions as rotations turn out to be an instance of a group acting on the objects in a geometry. In the next section, we will take up the same topic in the real projective plane.

Hamilton discovered the quaternions after an arduous search inspired by the beautiful connection he discovered between the complex numbers and the geometry of $\mathbb{R}^2$. Recall that the complex numbers, $\mathbb{C} = \{a+bi \mid a, b \in \mathbb{R}\}$ can be identified with the points in $\mathbb{R}^2 = \{(a,b) \mid a, b \in \mathbb{R}\}$. This is significant because it allows $\mathbb{C}$ and $\mathbb{R}^2$ to borrow structure from one another. To see this, note first that we can multiply elements of $\mathbb{C}$ simply using the distributive law and the definition of $i$ via $i^2 = -1$. Multiplication is commutative (easy to check) and associative (tedious to check.) Recall further that the *complex conjugate* of $z = a + bi \in \mathbb{C}$ is $\bar{z} = a - bi$. We then have $z\bar{z} = a^2 + b^2$, a nonnegative real number. This gives us the *modulus* or *norm* or *absolute value* of $z \in \mathbb{C}$, $\|z\| = \sqrt{a^2 + b^2}$. Notice that $\|z\|$ is the distance from the associated point in $\mathbb{R}^2$ to the origin. We define division in $\mathbb{C}$ by $z/w = z\bar{w}/\|w\|^2$. This makes sense whenever $w \neq 0$ and now we have that $\mathbb{C}$ is a field.

Consider the collection of *unit complex numbers* $\{z \in \mathbb{C} \mid \|z\| = 1\}$. Note that this is precisely the unit circle in $\mathbb{R}^2$, usually designated $S^1$. $S^1$ is the one sphere, that is, the sphere which is one dimensional. $S^1$ is not flat, it is not a vector space, so you don't have a precise idea of what "dimension" means for such an object. A safe way to think about it is as follows: the dimension of the object is the dimension of its tangent space. The tangent space to a circle is a line, which is one dimensional, therefore the dimension of $S^1$ is one.

If $z \in S^1$, then $z = \cos\theta + i\sin\theta$ for some $\theta$. In fact, it is an easy exercise in trigonometry to show that if $w \in \mathbb{C}$, $w = \|w\|(\cos\varphi + i\sin\varphi)$ . With $z$ and $w$ in hand, we have

$$z.w = \|w\|(\cos\theta\cos\varphi - \sin\theta\sin\varphi) + i(\sin\theta\cos\varphi + \cos\theta\sin\varphi) =$$

$$\|w\|(\cos(\theta+\varphi) + i\sin(\theta+\varphi)).$$

We see then that multiplication by a unit complex number effects a rotation on $\mathbb{R}^2$: it does not change the distance from a point to the origin but it does change its angle.

Hamilton wanted to extend this story to $\mathbb{R}^3$ and came up empty handed until he realized, at a famous moment when he was crossing a bridge in Dublin, that he needed to view $\mathbb{R}^3$ as sitting inside a four dimensional analog to $\mathbb{C}$, the quaternions $\mathbb{H}$.

Define $\mathbb{H} = \{q = a + bi + cj + dk \mid a, b, c, d \in \mathbb{R}\}$ where $i$, $j$, $k$ enjoy no relations among themselves except for the following and their consequences:

$$i^2 = j^2 = k^2 = -1$$

$$ij = k,\ ji = -k,\ jk = i,\ kj = -i,\ ki = j,\ ik = -j.$$

Using these relations and the distributive law, we can define multiplication of quaternions. Obviously multiplication is not commutative so $\mathbb{H}$ is not a field. It does, however, enjoy all the other properties of a field so is a *skew field.*

As in $\mathbb{C}$, we have conjugation in $\mathbb{H}$ given by $\bar{q} = a - bi - cj - dk$. Then $q\bar{q} = a^2 + b^2 + c^2 + d^2$ and the modulus of $q$ is $\|q\| = \sqrt{q\bar{q}}$. Again, this allows us to define division: $q/w = q\bar{w}/\|w\|^2$, for $q, w \in \mathbb{H}$.

$\mathbb{H}$ can be identified with $\mathbb{R}^4$. Making the identification, we write $q = a + bi + cj + dk = (a, b, c, d)$. The *pure quaternions* are those of the form $(0, b, c, d)$. Notice that we can designate any quaternion by $(a, \mathbf{v})$, $a \in \mathbb{R}$, $\mathbf{v} \in \mathbb{R}^3$. This is nice because it lets us write a rather involved looking formulation that actually helps us prove some results.

If $q = (a, \mathbf{v})$, and $w = (b, \mathbf{u})$ are in $\mathbb{H}$, then using the distributive law you should have no trouble verifying that

$$qw = (ab - \mathbf{v}\cdot\mathbf{u}, a\mathbf{u} + b\mathbf{v} + \mathbf{v}\times\mathbf{u}).$$

Here $\mathbf{v} \cdot \mathbf{u}$ is the usual dot product in $\mathbb{R}^3$, that is, for $\mathbf{v} = (v_1, v_2, v_3)$ and $\mathbf{u} = (u_1, u_2, u_3)$, $\mathbf{v} \cdot \mathbf{u} = v_1 u_1 + v_2 u_2 + v_3 u_3$. $\mathbf{v} \times \mathbf{u}$ is likewise the usual cross product in $\mathbb{R}^3$, where $i$, $j$, $k$ now play the roles of the standard unit vectors in $\mathbb{R}^3$.

$$\mathbf{v} \times \mathbf{u} = \det \begin{bmatrix} i & j & k \\ v_1 & v_2 & v_3 \\ u_1 & u_2 & u_3 \end{bmatrix}$$

We can identify $\mathbb{R}^3$ with the pure quaternions and then the unit pure quaternions become $S^2$, the 2-sphere, that is, the surface of a ball. The unit quaternions can be identified with $S^3$, the 3-sphere, an object that has inspired extensive research for over one hundred years.

Where are the rotations? Let $R_{\theta,\mathbf{v}}$ be the rotation of $\mathbb{R}^3$ about the axis determined by the unit vector $\mathbf{v}$, through the angle $\theta$. Notice that there are two choices for $\mathbf{v}$. If it makes things any easier we can always insist that if $\mathbf{v}$ points "up", then the rotation is counterclockwise.

The mapping $u \mapsto quq^{-1}$ is called *conjugation by $q$*. Conjugation comes up frequently in group theory and in geometry.

**Theorem 38.** *$R_{\theta,\mathbf{v}}$ is effected by conjugating any element of $\mathbb{R}^3$, viewed as a pure quaternion, by $(\cos(\theta/2), \sin(\theta/2).\mathbf{v})$.*

To prove/understand the theorem, the first thing we do is verify that conjugation maps pure quaternions to pure quaternions. For $u = (0, \mathbf{u})$ and $q = (a, \mathbf{v})$,

$$quq^{-1} = (a, \mathbf{v})(0, \mathbf{u})(a, -\mathbf{v}) =$$

$$(-\mathbf{v} \cdot \mathbf{u}, a\mathbf{u} + \mathbf{v} \times \mathbf{u})(a, -\mathbf{v}) = (-a\mathbf{v} \cdot \mathbf{u} + a\mathbf{v} \cdot \mathbf{u}, \dots) = (0, \dots)$$

where it doesn't matter what $\dots$ is, since we just wanted to show that $u$ is mapped to another pure quaternion.

An obvious check to whether or not $q = (\cos(\theta/2), \sin(\theta/2).\mathbf{v})$ really is a rotation of $\mathbb{R}^3$ with axis $< \mathbf{v} >$ is to see what $q\mathbf{v}q^{-1}$ gives us. Bearing in mind that $\mathbf{v}$ is a unit vector, we have $\mathbf{v} \cdot \mathbf{v} = 1$. Also, $\mathbf{v} \times \mathbf{v} = \mathbf{0}$, the zero vector. Then

$$q\mathbf{v}q^{-1} = (\cos(\theta/2), \sin(\theta/2).\mathbf{v})(0, \mathbf{v})(\cos(\theta/2), -\sin(\theta/2).\mathbf{v}) =$$

$$(-\sin(\theta/2), \cos(\theta/2).\mathbf{v})(\cos(\theta/2), -\sin(\theta/2).\mathbf{v}) =$$

$$(-\sin(\theta/2)\cos(\theta/2) + \sin(\theta/2)\cos(\theta/2), \cos^2(\theta/2).\mathbf{v} + \sin^2(\theta/2)\mathbf{v}) = (0, \mathbf{v}).$$

This proves that $\mathbf{v}$ is fixed under conjugation by $q$. This is promising, but not a proof.

To finish the proof, we think about a matrix representation of $R_{\theta,\mathbf{v}}$. Matrix representations depend on choice of ordered basis. A good basis to use here would be $\{\mathbf{v}, \mathbf{u}, \mathbf{w}\}$ where $\{\mathbf{v}, \mathbf{u}, \mathbf{w}\}$ form a *right-handed coordinate system* of *orthonormal vectors*. This means that all three basis elements are unit vectors, that they are orthogonal, in other words, have zero dot product, and that $\mathbf{v} \times \mathbf{u} = \mathbf{w}$, $\mathbf{u} \times \mathbf{w} = \mathbf{v}$, and $\mathbf{w} \times \mathbf{v} = \mathbf{u}$. In short, we can think of $\mathbf{v}$, $\mathbf{u}$, $\mathbf{w}$ as arranged like the unit vectors in the $x$, $y$, $z$ directions in $\mathbb{R}^3$. This allows us to visualize $R_{\theta,\mathbf{v}}$ as a counterclockwise rotation in the $yz$ plane. The matrix representation is then

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$$

To see this, think of where the unit vectors in the $y$ and $z$ direction are mapped under the rotation. In particular, we have

$$\mathbf{v} \mapsto \mathbf{v},\ \mathbf{u} \mapsto \cos\theta.\mathbf{u} + \sin\theta.\mathbf{w},\ \mathbf{w} \mapsto -\sin\theta.\mathbf{u} + \cos\theta.\mathbf{w}.$$

Now we check:

$$q\mathbf{u}q^{-1} = (\cos(\theta/2), \sin(\theta/2).\mathbf{v})(0, \mathbf{u})(\cos(\theta/2), -\sin(\theta/2).\mathbf{v}) =$$

$$(0, \cos(\theta/2).\mathbf{u} + \sin(\theta/2).\mathbf{w})(\cos(\theta/2), -\sin(\theta/2).\mathbf{v}) =$$

$$(0, \cos^2(\theta/2)\mathbf{u} + \cos(\theta/2)\sin(\theta/2)\mathbf{w} - \sin(\theta/2)\cos(\theta/2).\mathbf{u} \times \mathbf{v} - \sin^2(\theta/2)\mathbf{w} \times \mathbf{v}) =$$

$$0, (\cos^2(\theta/2) - \sin^2(\theta/2).\mathbf{u} + 2\sin(\theta/2)\cos(\theta/2).\mathbf{w}) = (0, \cos\theta.\mathbf{u} + \sin\theta.\mathbf{w}).$$

We leave it as an exercise to show that conjugation by $q$ maps $\mathbf{w} \mapsto -\sin\theta.\mathbf{u} + \cos\theta.\mathbf{w}$. That completes the proof of the theorem.

**Exercises**

1. Show that conjugation by $q \in \mathbb{H}$, $q \neq 0$, defines a linear transformation on $\mathbb{R}^3$.

2. In the proof of Theorem 38, why did we not have to show that every vector in $\mathbb{R}^3$ underwent a rotation?

3. What conditions would we have to impose on $q$ to get that conjugation by $q$ on pure quaternions preserves norm? In other words, when does $qwq^{-1}$ have the same norm as $w$, for $q \in \mathbb{H}$, $w$ pure quaternion?

4. Find the explicit expression for $q$ to effect the rotation in $\mathbb{R}^3$ through $30°$, about an axis that is normal to the plane $x + y + z = 0$. Take the rotation that is counterclockwise when viewed along the axis from the first octant. Use $q$ to determine where the point $(-1, 1, 2)$ is mapped to under the rotation. Give exact coordinates.

# 5 The Real Projective Plane

We consider here one model for the real projective plane, $\mathbb{R}P^2$.

**Theorem 39.** *Let $F$ be any field and consider the vector space $V = F^3$. By defining a* projective point *as a 1-dimensional subspace of $V$, a* projective line *as a 2-dimensional subspace of $V$, and* incidence *as nonempty intersection, we get a set, $\overline{\Pi}(V)$, that satisfies the axioms for a projective plane.*

*Proof.* We have four axioms to prove: the two that show nontriviality and the two incidence axioms. Since $F$ contains at least two elements, 0 and 1, we have the following distinct points in $\overline{\Pi}(V)$: $< (1, 0, 0) >$, $< (0, 1, 0) >$, $< (0, 0, 1) >$, and $< (1, 1, 1) >$. We claim that no three of these are collinear, that is, incident to the same line.

The line determined by $< (1, 0, 0) >$ and $< (0, 1, 0) >$ is the span of $\{(1, 0, 0), (0, 1, 0)\}$, that is, the set of all vectors in $V$ of the form $(\alpha, \beta, 0)$, $\alpha, \beta$ not both zero in $F$. Since this set contains neither $(0, 0, 1)$ nor $(1, 1, 1)$, neither of the latter points is collinear with $< (1, 0, 0) >$ and $< (0, 1, 0) >$. A similar argument applies to the other sets of three vectors. This verifies that $\overline{\Pi}(F)$ has four points, no three of which are collinear.

Next we consider a line in $\overline{\Pi}(F)$ and verify that it has three points. Every line in $\overline{\Pi}(F)$ is determined by a linearly independent set of vectors $\{u, v\} \subset F^3$. Note that the span of this set contains lines $< u >$, $< v >$ and $< u + v >$. It is easy to verify that these are distinct, thus, they correspond to three distinct points on the line determined by $u$ and $v$.

Next, let $\ell$ be determined by the span of $\{u, v\}$ and $\ell'$ be determined by the span of $\{w, x\}$. Since planes through the origin in $F^3$ must intersect in a unique line through the origin, $\ell$ and $\ell'$ intersect in a unique projective point. Similarly, if $< u >$ and $< v >$ determine distinct points in $\overline{\Pi}(F)$, then $\{u, v\}$ is a linearly independent set, thus spanning a 2-dimensional subspace of $F^3$, which determines the unique line in $\overline{\Pi}(F)$ determined by $< u >$ and $< v >$. $\square$

There are other models for $\mathbb{R}P^2$. One that can be useful for visualization is the unit sphere, $S^2$, with antipodal points identified. In other words, a point of $\mathbb{R}P^2$ is realized as the pair of endpoints of a diagonal of the unit sphere in $\mathbb{R}^3$. How is a line realized with this model? To understand this part, notice that for a pair of nonantipodal points on the sphere, there are many circles on the surface of the sphere that contain those points but there is only one great circle that contains those points. A way to think of this is to realize that you get a circle on the sphere precisely by intersecting $S^2$ with a plane. When you intersect $S^2$ with a plane through the origin, you get a great circle. If you pick two nonantipodal points on $S^2$ and the origin, this is a set of three noncollinear points, thus uniquely determines a plane through the origin. This plane, in turn, determines a unique great circle on $S^2$, which must then contain the antipodes of both original points. This great circle identifies the projective line determined by the two projective points.

This brief discussion shows right away how the $S^2$ model and the subspace model interact. In particular, a one dimensional subspace of $\mathbb{R}^3$ intersects $S^2$ in a unique pair of antipodal points. This is a bijective correspondence between points in the two models.

## 5.1 Homogeneous Coordinates

Designate a projective point now as $[u]$, that is, let $[x, y, z]$ be the projective point corresponding to the nonzero vector $u = (x, y, z)$. If $z \neq 0$, then $[u] = [x, y, z] = [x/z, y/z, 1]$. In particular, the coordinates of $[u]$ are defined only up to nonzero multiples, so are called *homogeneous coordinates.*

Homogeneous coordinates are especially convenient for *localizing*, that is, for thinking of points that are somehow "near" one another in $\mathbb{R}P^2$ as belonging to an affine space.

**Definition 20.** *The* line at infinity, *$\ell_\infty \subset \overline{\Pi}(F)$, is the projective line corresponding to the space $\{(a,b,0)|\ a,b \in F\}$.*

We leave the verification of the next theorem to the reader.

**Theorem 40.** *The mapping $\varphi : \Pi(F) \to \overline{\Pi}(F)$ given by $\varphi(a,b) = [a,b,1]$ is a bijection onto the set $\overline{\Pi}(F) \setminus \ell_\infty$.*

This gives us a visual aid for understanding $\Pi(F)$ as a subset of $\overline{\Pi}(F)$. The plane $z = 1$ intersects all lines through the origin in $F^3$ except those lying in the $xy$-plane. In other words, $< u >=< (a,b,c) >$ is a line in $F^3$ not lying in the $xy$-plane if and only if $c \neq 0$. Its intersection with the plane at $z = 1$ is the image of the affine point $(a/c, b/c)$ under $\varphi$. Now we can think of projective points $[a,b,1]$ as being near each other in the sense that they comprise an affine space.

Our choice of projective line to designate $\ell_\infty$ is arbitrary: we can excise any line from $\overline{\Pi}(F)$ to get a copy of $\Pi(F)$, that is a localization.

The first six exercises that follow are taken from [7].

**Exercises**

1. Describe the line in $\overline{\Pi}(\mathbb{R})$, determined by the following pairs of points.

   (a) $[1,1,2]$ and $[2,3-1]$

   (b) $[0,2,1]$ and $[-1,1,1]$

   (c) $[2,0,1]$ and $[-1,1,1]$.

   (d) $[1,2,0]$ and $[2,3,4]$

   (e) $[1,2,0]$ and $[2,3,0]$

2. Using the mapping described in the last theorem, find the preimage in $\Pi(\mathbb{R})$, if there is one, for each of the following projective points.

   (a) $[2,3,1]$

   (b) $[-2,-3,1]$

(c) $[-2, -3, -1]$
(d) $[2, 4, 6]$
(e) $[2, -3, -1]$
(f) $[-1, 2, 0]$
(g) $[1/2, 1/3, 6]$
(h) $[1, 0, 0]$

3. The following pairs of equations describe lines in $\mathbb{R}P^2$. Find the projective point where each pair intersects.

   (a) $x + y + z = 0$, and $-x + 2y + z = 0$
   (b) $2x - y + 3z = 0$ and $-x + z = 0$
   (c) $x = 0$ and $y = 0$
   (d) $x = 0$ and $z = 0$
   (e) $y = 0$ and $z = 0$
   (f) $-x + 5y - z = 0$ and $2x + 7y - 3z = 0$

4. Find the image of each of the following lines in $\Pi(\mathbb{R})$ under the mapping $\varphi$ given in the theorem above.

   (a) $x = 7$
   (b) $y = 8$
   (c) $x + y = 4$
   (d) $-x + 3y + 18 = 0$
   (e) $2x - 7y = 21$
   (f) $-y + 3x + 5 = 0$

5. Determine whether each of the following sets three projective lines is concurrent and find the points of concurrency.

   (a) $x + y + z = 0$, $2x - 3y + z = 0$, $-x + 7y - 6z = 0$
   (b) $x + 2y - z = 0$, $4x - y = 0$, $z = 0$
   (c) $2x - y + 3z = 0$, $-4x + 2y + z = 0$, $x - (1/2)y + z = 0$

6. Which of the following sets of projective points is collinear? For those that are, describe the line.
   (a) $[-1,-1,1]$, $[3,3,3]$, $[1/2,1/2,4]$
   (b) $[-2,3,0]$, $[1,-2,1]$, $[2,0,0]$
   (c) $[1,1,-1]$, $[2,0,-2]$, $[0,1,2]$
7. Prove that $\{[u],[v],[w]\} \subset \overline{\Pi}(F)$ is a triangle if and only if $\{u,v,w\}$ is a basis of $F^3$.

## 5.2 The Projective Group

Recall that $GL_3(F)$ is the group of nonsingular linear transformations from $F^3$ to itself. $GL_3(F)$ can be identified with $3 \times 3$ matrices over $F$ with nonzero determinant. At first blush, $GL_3(F)$ seems a natural choice for a group of bijections on $\overline{\Pi}(F)$ but we see trouble right away when we notice, for instance, that both $I_3$ and $-I_3$ fix all projective points. This means that on $\overline{\Pi}(F)$, $I_3$ and $-I_3$ define the same mapping. Indeed, for any $\alpha \in F$, $\alpha I_3$ is the identity mapping on $\overline{\Pi}(F)$. This suggests that we consider the following as the natural group of bijections on $\overline{\Pi}(F)$.

**Definition 21.** *The* projective group*, $PGL_3(F)$, is the collection of mappings $\bar{T} : \overline{\Pi}(F) \to \overline{\Pi}(F)$ where $\bar{T}([u]) := [T(u)]$ for $T \in GL_3(F)$.*

We put the cart before the horse with this definition: it really only makes any sense after we prove the following theorem.

**Theorem 41.** *If $T \in GL_3(F)$, then $\bar{T}$ defined by $\bar{T}[u] = [T(u)]$ is a well-defined bijective mapping on $\overline{\Pi}(F)$. Moreover, $\bar{I}_3$ is the identity mapping on $\overline{\Pi}(F)$, $\bar{T} \circ \bar{S} = \overline{T \circ S}$, , and $\bar{T}^{-1} = \overline{T^{-1}}$.*

*Proof.* We first must show that if $[u] = [v]$, then $\bar{T}[u] = \bar{T}[v]$ but that is immediate because $[u] = [v]$ if and only if there is nonzero $\alpha \in F$ with $v = \alpha u$, in which case

$$\bar{T}[u] = [T(u)] = [\alpha T(u)] = [T(\alpha u)] = [T(v)] = \bar{T}[v].$$

The statement about $\bar{I}_3$ is clear. Let $T$ and $S$ belong to $GL_3(F)$. For any $[u] \in \overline{\Pi}(F)$, we have

$$\bar{T} \circ \bar{S}[u] = \bar{T}[S(u)] = [T(S(u))] = [T \circ S(u)] = \overline{T \circ S}[u].$$

We leave the proof of the last statement to the reader. The fact that $\bar{T}$ is bijective follows. $\square$

**Theorem 42.** *The projective groups maps lines to lines and triangles to triangles in* $\overline{\Pi}(F)$.

*Proof.* A line $\ell \in \overline{\Pi}(F)$ corresponds to a two dimensional subspace $< u, v > \subset F^3$ and a mapping $\bar{T} \in PGL_3(F)$ corresponds to the collection of mappings $\alpha T \in GL_3(F)$, $\alpha$ nonzero in $F$, $T$ bijective. $T(\{u, v\})$ is a linearly independent set of two vectors hence corresponds to a two dimensional subspace $< T(u), T(v) > = < \alpha T(u), \alpha T(v) >$, that is, to a projective line $\ell' \in \overline{\Pi}(F)$. The argument for triangles is even easier, in light of Exercise 7 at the end of the last subsection. $\square$

Next we consider the relationship between the affine and the projective groups.

Recall that an element of Aff$(V)$ is $A(T, v)$, where $T \in GL(V)$, $v \in V$ and $A(T, v)(u) = T(u) + v$. Identifying $\Pi(F)$ with the plane $z = 1$ in $F^3$ lets us think of the elements of Aff$(V)$ as matrices with the form

$$A(T, v) = \left(\begin{array}{cc|c} a & b & x \\ c & d & y \\ \hline 0 & 0 & 1 \end{array}\right)$$

where $T = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(F)$ and $v = (x, y) \in F^2$. Using block matrix form, we can write

$$A(T, v) = \left(\begin{array}{c|c} T & v \\ \hline 00 & 1 \end{array}\right).$$

We then have

$$A(T, v) \circ A(T', v') = \left(\begin{array}{c|c} T & v \\ \hline 00 & 1 \end{array}\right) \left(\begin{array}{c|c} T' & v' \\ \hline 00 & 1 \end{array}\right) = \left(\begin{array}{c|c} TT' & T(v') + v \\ \hline 00 & 1 \end{array}\right).$$

Turning now to $PGL_3(F)$, we suppose $\bar{P}$ is a projective transformation that maps $\Pi(F)$ to itself. As usual, we identify $\Pi(F)$ with the plane $z = 1$ in $F^3$. It is not difficult to verify that$\bar{P}$ must be all nonzero scalar multiples of a matrix of the form

$$P = \left(\begin{array}{cc|c} & T & a \\ & & b \\ \hline 0 & 0 & 1 \end{array}\right)$$

where $\det P = \det T \neq 0$. In other words, we can represent $\bar{P}$ uniquely with a matrix of this form, that is, by an affine transformation.

**Theorem 43.** $\mathrm{Aff}(F^2)$ *can be identified as a subgroup of* $PGL_3(F)$ *that maps* $\ell_\infty$ *to itself and that fixes* $\Pi(F) \subset \overline{\Pi}(F)$*.*

**Exercises**

1. Finish the proof of Theorem 41 by showing that $\bar{T}^{-1} = \overline{T^{-1}}$.

2. Let $T$ linear be defined on $F^3$ by $T(1,0,0) = (0,1,0)$, $T(0,1,0) = (0,0,1)$ and $T(0,0,1) = (1,0,1)$.

   (a) Verify that $T \in GL_3(F)$.

   (b) Find $\bar{T}[1,2,-1]$.

   (c) Find $\bar{T}(\ell_\infty)$.

   (d) Does $\bar{T} : \Pi(F) \to \Pi(F)$? If so, prove it. If not, describe $\bar{T}(\Pi(F))$.

3. Complete the proof of Theorem 42 by showing that $PGL_3(F)$ maps triangles to triangles. Show that if we consider a triangle an ordered triple of noncollinear points, then there is exactly one element in the projective group that sends a given triangle to another.

4. Verify that$\bar{P} \in PGL_3(F)$ maps $\Pi(F)$ to itself if and only if $\bar{P}$ corresponds to the set of all nonzero scalar multiples of a matrix of the form

$$P = \left(\begin{array}{cc|c} & T & a \\ & & b \\ \hline 0 & 0 & 1 \end{array}\right)$$

   for some $T \in GL_2(F)$.

5. Verify that a mapping of the form given in the last problem fixes $\ell_\infty$.

## 5.3 Curves

Curves are a source of much comment in calculus. Typically, in that context, you study pieces of curves, in particular, the pieces that can be described as graphs of functions, $y = f(x)$, for example, $y = \sqrt{x}$. More generally, we define a real algebraic plane curve, that is, a curve in $\mathbb{R}^2$, as the zero set of a polynomial in $x$ and $y$. Instead of thinking of functions, we are thinking of polynomial equations. The equation associated to the function $y = \sqrt{x}$ would be $y^2 = x$ or $x - y^2 = 0$.

A *homogeneous polynomial of degree* $n$ is a sum of monomials of degree $n$. $x - y^2$ is an example of an *inhomogeneous polynomial* of degree 2. Note that the degree of the polynomial is that of the highest degree monomial.

In $\overline{\Pi}(F)$, an algebraic curve is the set of zeroes of a homogeneous polynomial. We can understand curves in $F^2$ by "lifting" to $\overline{\Pi}(F)$ through the process of homogenization. To homogenize a polynomial in $x$ and $y$ of degree $n$, multiply each monomial by $z^{d-n}$, where $d$ is the degree of the monomial. We homogenize a polynomial equation in $x$ and $y$ by first writing it in the form $p(x, y) = 0$, then homogenizing $p(x, y)$.

**Example 5.** *If we homogenize $x - y^2$, we get $xz - y^2$. To homogenize $y = x^2 - 2$, rewrite to get $x^2 - y - 2 = 0$ then homogenize the left hand side to get $x^2 - yz - 2z^2 = 0$.*

**Exercises**

1. Consider the equation $2x + 3y = 1$.
   (a) What does this equation describe in $\Pi(\mathbb{R})$?
   (b) Homogenize the equation.
   (c) What does the homogenized equation describe in $\mathbb{R}P^2$?
2. Consider the set of points in $\mathbb{R}^2$ given by $(t, t^2/2)$.
   (a) What does this set of points describe?
   (b) Write an equation that gives a relation satisfied by the $x$ and $y$ coordinates of the points on this curve.
   (c) Homogenize the equation.

(d) Describe the set of points in $\overline{\Pi}(\mathbb{R})$ that satisfy the homogeneous equation using parameters $s$ and $t$. (Hint: Think of embedding the points $(t, t^2/2)$ into $\mathbb{R}^3$ by putting them in the plane $z = 1$. Now homogenize the resulting parametric description.)

## 5.4 Conics

Recall that in $\mathbb{R}^2$, we can define a conic section to be the set of zeroes of any degree two polynomial.

$$ax^2 + by^2 + cxy + dx + ey + f.$$

When the constant term, $f$, is not zero, and at least one of $a$, $b$, $c$ is not zero, the conic is nondegenerate, that is, it is a bona fide parabola, hyperbola or ellipse, as opposed to one or two lines or a single point. If we homogenize our defining polynomial, we get

$$ax^2 + by^2 + cxy + dxz + eyz + fz^2$$

It may be hard to see what this defines. Note that we can get rid of cross terms by rotations, though. To help you believe that, we note that if we send $x \mapsto x + y$ and $y \mapsto x - y$ then $xy \mapsto x^2 - y^2$. Now look at the $xy$-plane, and compare the $x$ axis to the line $x + y$, and the $y$ axis to the line $x - y$: we rotate through an angle of 45° to effect the transformation. This lets us see that we lose no generality if we consider that the defining polynomial for a conic in $\mathbb{R}P^2$ is given by $ax^2 + by^2 + fz^2$. Next note that scaling in the $x$, $y$, $z$ directions separately is defined by a diagonal matrix which determines a projective transformation. To understand how to think of a conic in $\mathbb{R}P^2$, all we care about are the zeroes of the polynomial. It follows that the only thing that matters here is the ratio of signs of the coefficients of $x^2$, $y^2$, and $z^2$. This means that we can think of a conic section in the projective plane as associated to the surface given by $x^2 + y^2 - z^2 = 0$, that is, a cone. Different conic sections then arise via a transformation of the cone that amounts to a rotation about the origin in $\mathbb{R}^3$ or scaling in various directions, neither of which change the cone into a different type of surface. This makes the following theorem transparent.

**Theorem 44.** *Under projective transformations, all (nondegenrate) conic sections are the same. In other words, given any two conic sectionsin $\mathbb{R}P^2$, there is a projective transformation that sends one to the other.*

## 5.5 Degree

Our goal here is to show that the degree of a curve in $\bar{\Pi}(F)$ is actually a geometric feature of the curve, not just an attribute of one polynomial that may define the curve. We need a few results about homogeneous polynomials to get there. This material draws from Walker's classic book on algebraic curves [10].

**Theorem 45.** *The factors of a homogeneous polynomial are homogeneous.*

*Proof.* Suppose $F = fg$, where $F$ is homogeneous and $f$ is not homogeneous. Write $f$ and $g$ using homogeneous components as follows:

$$f = F_i + \ldots + F'_{i+j} \; g = G_k + \ldots + G_{k+\ell}$$

where $F_i$, $F_{i+j}$ are nonzero,homogeneous with degrees respectively $i$ and $i+j$, and $G_k$, $G_{k+\ell}$ are nonzero homogeneous of degrees $k$ and $k+\ell$. Attempting to write $fg$ using homogeneous components we get

$$fg = F_iG_k + (F_iG_{k+1} + F_{i+1}G_k) + \ldots + F_{i+j}G_{k+\ell}$$

where $F_iG_k$ and $F_{i+j}G_{k+\ell}$ are both nonzero with $\mathrm{degF_iG_k = i{+}k < degF_{i+j}G_{k+\ell}}$, contradicting $F$ homogeneous. □

Let $F_n$ designate a homogeneous polynomial of degree $n$. $F_n$ can have any number of variables. Say it has $k$ variables, that is, $F_n = F_n(x_1, \ldots, x_k)$. Let $f$ be the nonhomogeneous polynomial in $k-1$ variables that we get by letting $x_i = 1$, for some $i \in \{1, \ldots, k\}$. We say $F_n$ and $f$ are *associates.*

The next three results, variations on a theme, follow immediately from the theorem.

**Corollary 5.** *The factors of $F_n$ are associates of the factors of $f$ where $F_n$ and $f$ are associates.*

**Corollary 6.** *$F$ is irreducible if and only if its associates are irreducible.*

A field is *algebraically closed* provided all polynomials in a single variable with positive degree factor into linear factors. Our favorite example of an algebraically closed field is $\mathbb{C}$. Note that $\mathbb{R}$ is not algebraically closed as, for example, $x^2 + 1$ does not factor in linear factors if we restrict to coefficients in $\mathbb{R}$.

**Corollary 7.** *Let $D$ be algebraically closed and $F_n$ homogeneous degree $n$ in $D[x_0, x_1]$. There are $a \neq 0$, $a_i$, $b_i$ in $D$ with $F = a\Pi_{i=1}^{n}(a_i x_1 - b_i x_0)$. In other words, $F$ factors into linear factors.*

Now consider an algebraic curve in $\bar{\Pi}(F)$. This is the set of projective points that satisfy $F_n(x, y, z) = 0$, where $F_n(x, y, z)$ is homogeneous of degree $n$. Let $\ell$ be any line in $\bar{\Pi}(F)$. Think of $\ell$ as determined by a two dimensional subspace spanned by vectors $\mathbf{u}$ and $\mathbf{v}$ in $F^3$. We can think of $\ell$ as the set of all nonzero points $s\mathbf{u}+t\mathbf{v}$, where $s, t \in F$. In other words, $[s, t]$ are homogeneous coordinates for a point on $\ell$.

Consider what it means for a point to lie on both $C$ and $\ell$. Because it lies on $\ell$, it is $s\mathbf{u} + t\mathbf{v}$, for some $(s, t) \in F^2 \setminus (0, 0)$. Because it lies on $C$, $F_n(s\mathbf{u} + t\mathbf{v}) = 0$. A point of intersection of $C$ and $\ell$ then corresponds precisely to a solution $[s, t]$ to $F_n(s\mathbf{u}+t\mathbf{v}) = 0$. We are now thinking of $s$ and $t$ as variables so that $F_n(s\mathbf{u} + t\mathbf{v})$ is homogeneous degree $n$ in two variables. Our corollary above allows us to say that, counting multiplicity, there are $n$ solutions to $F_n(s\mathbf{u} + t\mathbf{v}) = 0$ in $\bar{\Pi}(F)$, as long as $F$ is algebraically closed. This proves the following theorem.

**Theorem 46.** *Assume $F$ is algebraically closed. The degree of a curve in $\bar{\Pi}(F)$ is the number of intersections it has with a line in $\bar{\Pi}(F)$*

The definition of degree as a geometric property of a curve is the first subplot in the story of intersections and singular points of curves. An important tool in the telling is *resultants*. We end these notes with a brief discussion of this idea.

**Definition 22.** *Let $f(x) = a_0 + a_1x + a_2x^2 + \ldots + a_nx^n$, $g(x) = b_0 + b_1x + b_2x^2 + \ldots + b_mx^m$, $a_i, b_i \in F$, $a_n, b_m \neq 0$. The* resultant, *$R(f, g)$ is the determinant*

$$\begin{vmatrix} a_0 & a_1 & \ldots & & a_n & & \\ 0 & a_0 & a_1 & \ldots & a_{n-1} & a_n & \\ & & & & & & \\ \vdots & & & & & & \\ b_0 & b_1 & \ldots & b_{m-1} & b_m & & \\ & & & & & & \\ \vdots & & & & & & \\ & & & 0 & b_0 & \ldots & b_m \end{vmatrix}$$

*where there are $m$ rows of $a_i$s and $n$ rows of $b_i$s.*

Note that the resultant is the determinant of an $m+n \times m+n$ matrix.

**Example 6.** *Let* $f(x) = x^2 - 2x + 1$ *and let* $g(x) = x - 1$. *Then*

$$R(f,g) = \begin{vmatrix} 1 & -2 & 1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{vmatrix} = 0.$$

The resultant of a polynomial and its first derivative is called the *discriminant* of the polynomial. We are used to seeing the discriminant defined for a quadratic polynomial $f(x) = ax^2 + bx + c$ as $b^2 - 4ac$. If you think about what the discriminant gives us, you start to get a sense of what resultants are good for more generally. In particular, when the discriminant of a quadratic polynomial is zero, the polynomial has a repeated root, that is, a root in common with its derivative. A quick calculation reveals that the definition we have here defines the discriminant of $f(x) = ax^2 + bx + c$ as $-a(b^2 - 4ac)$. Obviously, this is zero if and only if $b^2 - 4ac = 0$. As is typically the case with determinants, we only care whether the resultant is zero or not zero so for what we need, the two definitions are close enough.

**Theorem 47.** $R(f,g) = 0$ *if and only if* $f$ *and* $g$ *have a common nonconstant factor.*

*Proof.* We sketch the proof.

Start by noting that $f$ and $g$ have a nonconstant common factor if and only if there are $\varphi(x)$ and $\psi(x)$ with degree $\varphi(x) <$ degree $g(x)$, degree $\psi(x) <$ degree $f(x)$ and $f\varphi = g\psi$. Now think of $\varphi(x) = \alpha_0 + \alpha_1 x + \ldots + \alpha_k x^k$, $\psi(x) = \beta_0 + \beta_1 x + \ldots + \beta_\ell x^\ell$, where $\alpha_i$, $\beta_i$ are unknowns. After doing the multiplications to write $f\varphi = g\psi$, we see that we have equations giving us $a_0\alpha_0 = b_0\beta_0$, $a_0\alpha_1 + a_1\alpha_0 = b_0\beta_1 + b_1\beta_0$, etc. Essentially, the resultant is the determinant of the (transpose of the) coefficient matrix for this large homogeneous system of equations. The system has a nontrivial solution— $\alpha_i$s, $\beta_i$s not all zero— precisely when the determinant of this matrix, the resultant, is zero. □

In the projective plane, an algebraic curve is defined by a homogeneous polynomial in three variables. Two such polynomials with a nonconstant common factor define curves with a common component. How can we use resultants to pick up on these common components?

Let $f(x_1, \dots, x_k)$ be a polynomial with coefficients in a field $F$. We can think of $f$ as belonging to $D[x_k]$, where $D = F[x_1, \dots, x_{k-1}]$. $D$ is an example of an *algebra*, that is, a vector space that is closed under multiplication.

**Example 7.** *Let $f(x, y, z) = 2xy - x^2 + z^2$. We can think of $f$ as belonging to $D[x]$, $D = \mathbb{R}[y, z]$. Then we have $f = z^2 - (2y)x - x^2$, so the "constant" term is $z^2$, the coefficient of $x$ is $-2y$ and the coefficient of $x^2$ is $-1$.*

This idea allows us to define resultants for polynomials in several variables. A resultant for $f(x_1, \dots, x_k)$ and $g(x_1, \dots, x_k)$ is then a polynomial in $k - 1$ variables.

**Theorem 48.** *If $F_n, G_m$ are homogeneous of degrees $n$, $m$ in $F[x_0, x_1]$, then $R(F_n, G_m) = 0$ if and only if $F_n$ and $G_m$ have a common nonconstant factor.*

**Exercises**

1. Find the intesection points of $f(x, y, z) = x^2 + y^2 - z^2$ and the line $x + y + z = 0$.

2. Find the resultant of the two polynomials from the last exercise. (Note that there are three possibly different resultants. Find all of them.)

3. Using the definition of discriminant given in the text, find the discriminant of $x^3 + px + q$.

4. Consider the determinant

$$\begin{vmatrix} a_0 & a_1 & a_2 & \dots & a_{n-1} & a_n \\ -\lambda & 1 & 0 & \dots & 0 & 0 \\ 0 & -\lambda & 1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & -\lambda & 1 \end{vmatrix}.$$

   Argue that this must be equal to $a_0 + a_1\lambda + a_2\lambda^2 + \ldots + a_n\lambda^n$.

5. How can you tell, without doing the calculation, what the resultant of $f(x, y) = x^2 - 2xy + y^2$ and $g(x, y) = x - y$ is? Now do the calculation.

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